Analytic Calculi for Substructural Logics: Theory and Applications

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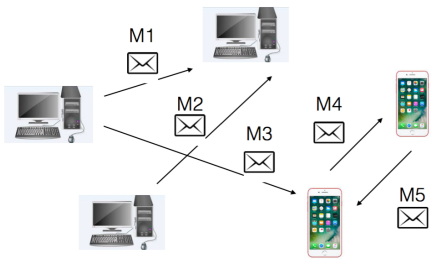
Proof Theory & Universal Algebra

Computation



Substructural logics

- include
  - intuitionistic logic,
  - intermediate logics,
  - relevance logics,
  - linear logic,
  - fuzzy logics,
  - ...
- lack the properties expressed by sequent calculus structural rules
- useful for reasoning about natural language, vagueness, resources, dynamic data structures, algebraic varieties, *concurrency* ...

This Talk

Theory

- Systematic and automated introduction of sequent and hypersequent calculi
  with *Kazushige Terui & Nikolaos Galatos*
  (LICS 2008, Algebra Universalis 2011, APAL 2012, APAL 2017)

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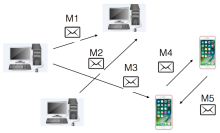
Chapter I

Applications



- Extraction of concurrent λ-calculi
  with *Federico Aschieri* & *Francesco A. Genco*

  (LICS 2017, Submitted 2018)

Chapter III

- From hypersequent calculi to natural deduction systems
  with *Francesco A. Genco*

  (TOCL 2018)

Chapter II

Substructural Logics

Substructural logics

Defined as

- *axiomatic* extensions of Full Lambek calculus **FL**
- *subvarieties* of (pointed) residuated lattices RL

Algebraization

For any set $\mathcal{A} \cup \{A, B\}$ of formulas,

$$\mathcal{A} \vdash_{\mathbf{FL}+B} A \text{ iff } \varepsilon[\mathcal{A}] \vDash_{RL+\varepsilon(B)} \varepsilon(A)$$

where $\varepsilon(-)$ is the equation corresponding to $-$.

Example: Gödel logic

obtained by adding

$$(\alpha \to \beta) \vee (\beta \to \alpha)$$

to intuitionistic logic (**FL** + exchange, weakening and contraction)
or equivalently

$$1 \leq (x \to y) \vee (y \to x)$$

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or equivalently

$$1 \leq (x \to y) \vee (y \to x)$$

to Heyting algebras (RL + commutativity, integrality and idempotency)

Why analytic calculi?

Substructural logics

Defined as

- *axiomatic* extensions of Full Lambek calculus **FL**
- *subvarieties* of (pointed) residuated lattices RL

Their applicability/usefulness strongly depends on the availability of

Analytic calculi

- useful for establishing various properties of logics
- key for developing automated reasoning methods

Sequent Calculus

Sequents (Gentzen 1934)

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$$

Axioms: E.g. $A \Rightarrow A, \quad \bot \Rightarrow A$

Rules:

- Structural
  E.g.

$$\frac{\Gamma, B, A \Rightarrow \Pi}{\Gamma, A, B \Rightarrow \Pi}\,(e,l) \quad \frac{\Gamma, A, A \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\,(c,l) \quad \frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\,(w,l)$$

- Logical (left and right)
- Cut

$$\frac{\Gamma \Rightarrow A \quad \Sigma, A \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Pi}\,Cut$$

Sequent Calculus

Sequents (Gentzen 1934)

$$A_1, \ldots, A_n \Rightarrow B$$

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- Logical (left and right)
- Cut

$$\frac{\Gamma \Rightarrow A \quad \Sigma, A \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Pi}\,Cut$$

Sequent Calculus – the cut rule

$$\boxed{\frac{\Gamma \Rightarrow A \quad A \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Pi}\ Cut}$$

- key to prove completeness w.r.t. Hilbert systems

$$\text{modus ponens} \quad \frac{A \quad A \to B}{B}$$

- bad for proof search

Cut-elimination theorem

Each proof using Cut can be transformed into a proof without Cut.

Sequent Calculus – state of the art

Cut-free sequent calculi have been successfully used

- to prove consistency, decidability, interpolation, ...
- to give syntactic proofs of algebraic properties for which (in particular cases) semantic methods are not known or do not work well

Many useful and interesting logics have no cut-free sequent calculus

Some extensions of the sequent calculus

A large range of generalizations of sequent calculus have been introduced

- hypersequent calculus (Avron, Mints, Pottinger)
- display calculus (Belnap)
- nested sequents (Brünnler, Fitting)
- deep inference (Guglielmi)
- bunched calculi (Dunn, Mints, . . .)
- labelled systems (Gabbay, Negri, Viganó, . . .)
- systems of rules (Negri)
- many placed sequents (TU Vienna)
- ...

Defining analytic calculi: state of the art

The definition of analytic calculi is usually logic-tailored.

Steps:

(i) choosing a framework
(ii) looking for the "right" inference rule(s)
(iii) proving cut-elimination

Systematic introduction of (hyper)sequent calculi

Chapter I

The base logic: **FLe**

- FLe ≈ commutative Full Lambek calculus
- FLe ≈ intuitionistic logic without weakening and contraction
- FLe ≈ intuitionistic Linear Logic without exponentials

Algebraic semantics:
A *(bounded pointed) commutative residuated lattice* is

$$\mathbf{P} = \langle P, \wedge, \vee, \otimes, \rightarrow, \top, \mathbf{0}, \mathbf{1}, \bot \rangle$$

1. $\langle P, \wedge, \vee \rangle$ is a lattice with $\top$ greatest and $\bot$ least
2. $\langle P, \otimes, \mathbf{1} \rangle$ is a commutative monoid.
3. For any $x, y, z \in P$, $x \otimes y \leq z \Longleftrightarrow y \leq x \rightarrow z$
4. $\mathbf{0} \in P$.

Sequent calculus for **commutative FL**

FLe

$$\frac{A, B, \Gamma \Rightarrow \Pi}{A \otimes B, \Gamma \Rightarrow \Pi} \otimes l \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B} \otimes r$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{\Gamma, A \to B, \Delta \Rightarrow \Pi} \to l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B} \to r$$

$$\frac{A, \Gamma \Rightarrow \Pi \quad B, \Gamma \Rightarrow \Pi}{A \vee B, \Gamma \Rightarrow \Pi} \vee l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \vee r \qquad \frac{}{\mathbf{0} \Rightarrow} \mathbf{0}l$$

$$\frac{A_i, \Gamma \Rightarrow \Pi}{A_1 \wedge A_2, \Gamma \Rightarrow \Pi} \wedge l \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \wedge r \qquad \frac{}{\Gamma \Rightarrow \top} \top r$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathbf{0}} \mathbf{0}r \qquad \frac{}{\Rightarrow \mathbf{1}} \mathbf{1}r \qquad \frac{}{\bot, \Gamma \Rightarrow \Pi} \bot l \qquad \frac{\Gamma \Rightarrow \Pi}{\mathbf{1}, \Gamma \Rightarrow \Pi} \mathbf{1}l$$

(Commutative) Substructural Logics

defined by adding Hilbert axioms to the sequent calculus **FLe** (or algebraic equations to commutative residuated lattices).

From axioms to rules: example

- Contraction: $\alpha \to \alpha \otimes \alpha$ $\quad \dfrac{A, A, \Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}\ (c)$
- Weakening l: $\alpha \to 1$ $\quad \dfrac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (w, l)$
- Weakening r: $0 \to \alpha$ $\quad \dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (w, r)$

Equivalence between rules and axioms

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- Weakening r: $0 \to \alpha$ $\quad \dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (w, r)$

Equivalence between rules and axioms

$$\vdash_{FLe+(axiom)} \quad = \quad \vdash_{FLe+(rule)}$$

For which axioms can we do it?

Algebraic Proof Theory

(AC, N. Galatos and K. Terui – APAL 2012, APAL 2017)

- Which Hilbert axioms can be transformed into rules that preserve cut-elimination?
- Which algebraic equations over residuated lattices are preserved by algebraic completions?

A completion of an algebra **A** is a complete algebra **B** (i.e. it has arbitrary $\bigvee$ and $\bigwedge$) such that $\mathbf{A} \subseteq \mathbf{B}$.

Classification

Formulas are **classified** according to the

polarity of their connectives w.r.t. a calculus (e.g., **FLe**)

(J.-M. Andreoli, 1992)

- *Positive* polarity: rule introducing the connective on the *left* is invertible

$$\text{E.g.} \quad \frac{\Gamma, A \Rightarrow \Pi \quad \Gamma, B \Rightarrow \Pi}{\Gamma, A \vee B \rightarrow \Pi} \vee l$$

- *Negative* polarity: rule introducing the connective/quantifier on the *right* is invertible

$$\text{E.g.} \quad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \rightarrow r$$

Substructural Hierarchy

Definition (AC, Galatos and Terui, LICS 2008)

The classes $\mathcal{P}_n, \mathcal{N}_n$ of positive and negative $\boxed{\text{axioms/equations}}$ are:

- $\mathcal{P}_0 ::= \mathcal{N}_0 ::=$ atomic formulas
- $\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \otimes \mathcal{P}_{n+1} \mid 1 \mid \bot$
- $\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0 \mid \top$

$$
\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \cdots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r] \arrow[ur] & \mathcal{N}_4 \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}
$$

Examples

Class	Axiom	Name
$\mathcal{N}_2$	$\alpha \to \mathbf{1}$, $\bot \to \alpha$	weakening
	$\alpha \to \alpha \otimes \alpha$	contraction
	$\alpha \otimes \alpha \to \alpha$	expansion
	$\otimes\alpha^n \to \otimes\alpha^m$	knotted axioms
	$\neg(\alpha \wedge \neg\alpha)$	weak contraction
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	excluded middle
	$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\mathcal{P}_3$	$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
	$\neg(\alpha \otimes \beta) \vee (\alpha \wedge \beta \to \alpha \otimes \beta)$	(wnm)
$\mathcal{N}_3$	$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$	Łukasiewicz axiom
	$(\alpha \wedge \beta) \to \alpha \otimes (\alpha \to \beta)$	divisibility

The hierarchy collapses to the class $\mathcal{N}_3$ (Jerabek 2016, commutative case)

Algorithm: from axioms to rules

Ingredient 1

The use of the invertible logical rules of **FLe**

Ingredient 2: Ackermann Lemma

An algebraic equation $t \leq u$ is equivalent to a quasiequation $u \leq x \Rightarrow t \leq x$, and also to $x \leq t \Rightarrow x \leq u$, where x is a fresh variable not occurring in t, u.

Example: the sequent $A \Rightarrow B$ is equivalent to

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow B} \qquad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}$$

($\Gamma(\Delta)$ fresh metavariable for multisets of formulas (at most one))

From Axioms to Sequent Rules

Algorithm to transform axioms/equations up to the class $\mathcal{N}_2$:

- into "good" structural rules in sequent calculus

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \cdots \\
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\end{tikzcd}$$

- into "good" quasiequations

$$t_1 \leq u_1 \text{ and..and } t_m \leq u_m \Rightarrow t_{m+1} \leq u_{m+1}$$

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Dedekind Completion of Rationals

- For any $X \subseteq \mathbb{Q}$,

$$\begin{aligned} X^{\triangleright} &= \{y \in \mathbb{Q} : \forall x \in X . x \leq y\} \\ X^{\triangleleft} &= \{y \in \mathbb{Q} : \forall x \in X . y \leq x\} \end{aligned}$$

- X is closed if $X = X^{\triangleright\triangleleft}$
- $(\mathbb{Q}, +, \cdot)$ can be embedded into $(\mathcal{C}(\mathbb{Q}), +, \cdot)$ with

$$\mathcal{C}(\mathbb{Q}) = \{X \subseteq \mathbb{Q} : X \text{ is closed}\}$$

From Axioms to Sequent Rules

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Beyond $\mathcal{N}_2$?

$$\text{Ex.} \quad (\alpha \to \beta) \vee (\beta \to \alpha)$$

Limitative Result

Analytic sequent calculi $\Longrightarrow$ Dedekind-MacNeille completion

(AC, N. Galatos and K. Terui. APAL 2012)

Hypersequent calculus

Axioms within the class $\mathcal{P}_3$ have the form

$$\mathcal{N}_2 \vee \mathcal{N}_2 \vee \cdots \vee \mathcal{N}_2$$

Hypersequents

$$\Gamma_1 \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Pi_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \Rightarrow \Pi_i$ is a sequent.

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Hypersequent calculus

It is obtained embedding sequents into hypersequents

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$$\frac{G \,|\, \Gamma \Rightarrow A \quad G \,|\, A, \Delta \Rightarrow \Pi}{G \,|\, \Gamma, \Delta \Rightarrow \Pi}\ Cut \qquad \frac{}{G \,|\, A \Rightarrow A}\ Identity$$

$$\frac{G \,|\, \Gamma \Rightarrow A \quad G \,|\, B, \Delta \Rightarrow \Pi}{G \,|\, \Gamma, A \to B, \Delta \Rightarrow \Pi} \to l \qquad \frac{G \,|\, A, \Gamma \Rightarrow B}{G \,|\, \Gamma \Rightarrow A \to B} \to r$$

and adding suitable rules to manipulate the additional layer of structure.

$$\frac{G \,|\, \Sigma \Rightarrow B \,|\, \Gamma \Rightarrow A \,|\, G'}{G \,|\, \Gamma \Rightarrow A \,|\, \Sigma \Rightarrow B \,|\, G'}\ (\mathrm{ee}) \qquad \frac{G}{G \,|\, \Gamma \Rightarrow A}\ (\mathrm{ew}) \qquad \frac{G \,|\, \Gamma \Rightarrow A \,|\, \Gamma \Rightarrow A}{G \,|\, \Gamma \Rightarrow A}\ (\mathrm{ec})$$

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Hypersequent calculus: an example

Gödel logic = Intuitionistic logic + $(\alpha \to \beta) \vee (\beta \to \alpha)$

$$\frac{G \,|\, \Gamma, \Sigma' \Rightarrow \Delta' \quad G \,|\, \Gamma', \Sigma \Rightarrow \Delta}{G \,|\, \Gamma, \Sigma \Rightarrow \Delta \,|\, \Gamma', \Sigma' \Rightarrow \Delta'}\ (com)$$

(Avron, Annals of Math and art. Intell. 1991)

$$\dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta \qquad \alpha \Rightarrow \alpha}{\alpha \Rightarrow \beta \,|\, \beta \Rightarrow \alpha}\ (\mathrm{com})}{\Rightarrow \alpha \to \beta \,|\, \Rightarrow \beta \to \alpha}\ (\to,\mathrm{r}),(\to,\mathrm{r})}{\Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha) \,|\, \Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\ (\vee_{\mathrm{i}},\mathrm{r}),(\vee_{\mathrm{i}},\mathrm{r})}{\Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\ (\mathrm{EC})$$

From Axioms to Hypersequent Rules

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\end{tikzcd}$$

Algorithm to transform axioms/equations up to the class $\mathcal{P}_3$:

- into "good" structural rules in hypersequent calculus
- into "good" analytic clauses

$$t_1 \leq u_1 \text{ and..and } t_m \leq u_m \Rightarrow t_{m+1} \leq u_{m+1} \text{ or..or } t_n \leq u_n$$

Limitative Result

Analytic hypersequent calculi $\Longrightarrow$ hyperDedekind-MacNeille completion

Example

$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$$

From Axioms to Hypersequent Rules

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Example

$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$$

From axioms to rules – our tool

https://www.logic.at/tinc/webaxiomcalc

Input: Hilbert axioms

TINC - AxiomCalc

General Information | AxiomCalc Web Interface | Obtaining AxiomCalc | Using AxiomCalc | References | Contact

Input Syntax

Examples

Input Axiom: Check for Standard Completeness

Check

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Output

Output AxiomCalc for: (a -> b) v (b -> a)

```
This axiom is in the class: p(2)


Equivalent Analytic Rule:

G|G+1,D+2 => P+2  G|G+2,D+1 => P+1
-----------------------------
 G| G+2,D+2 => P+2| G+1,D+1 => P+1
```

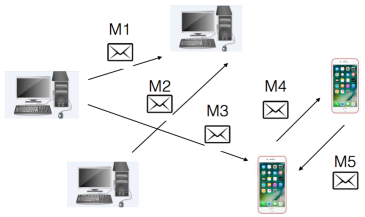
Get a paper containing an analytic calculus for FLe/HFLew extended with this axiom!

Ok

"Applications"

- closure under algebraic completions of large classes of equations
- decidability results
- standard completeness proofs (i.e. completeness of axiomatic systems w.r.t. algebras whose lattice reduct is the real unit interval $[0, 1]$)
- ...

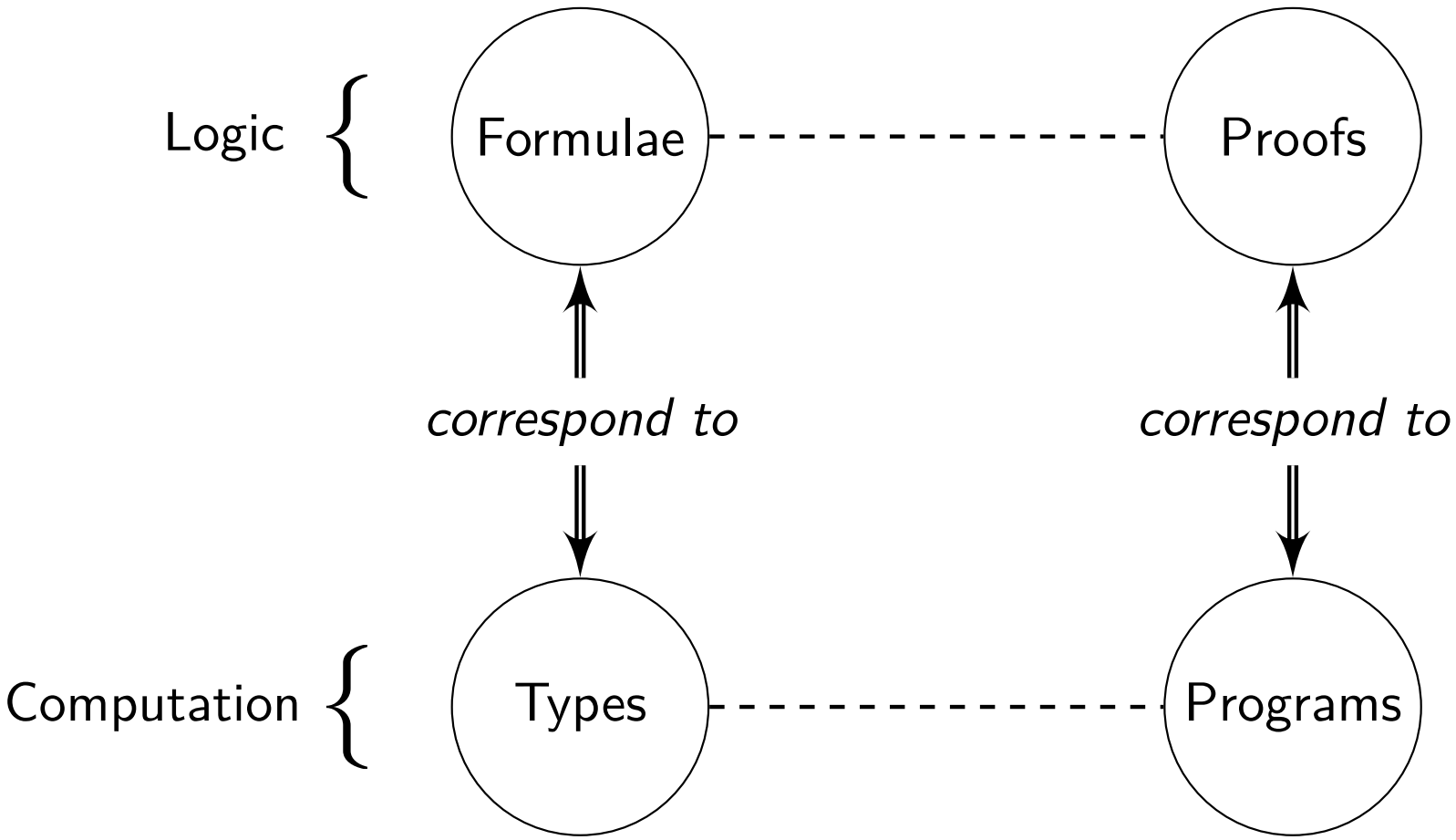
From Proof Theory to Computation



"We believe that logics with a cut-free hypersequent calculus could serve as bases for parallel λ-calculi."
(Avron 1991)

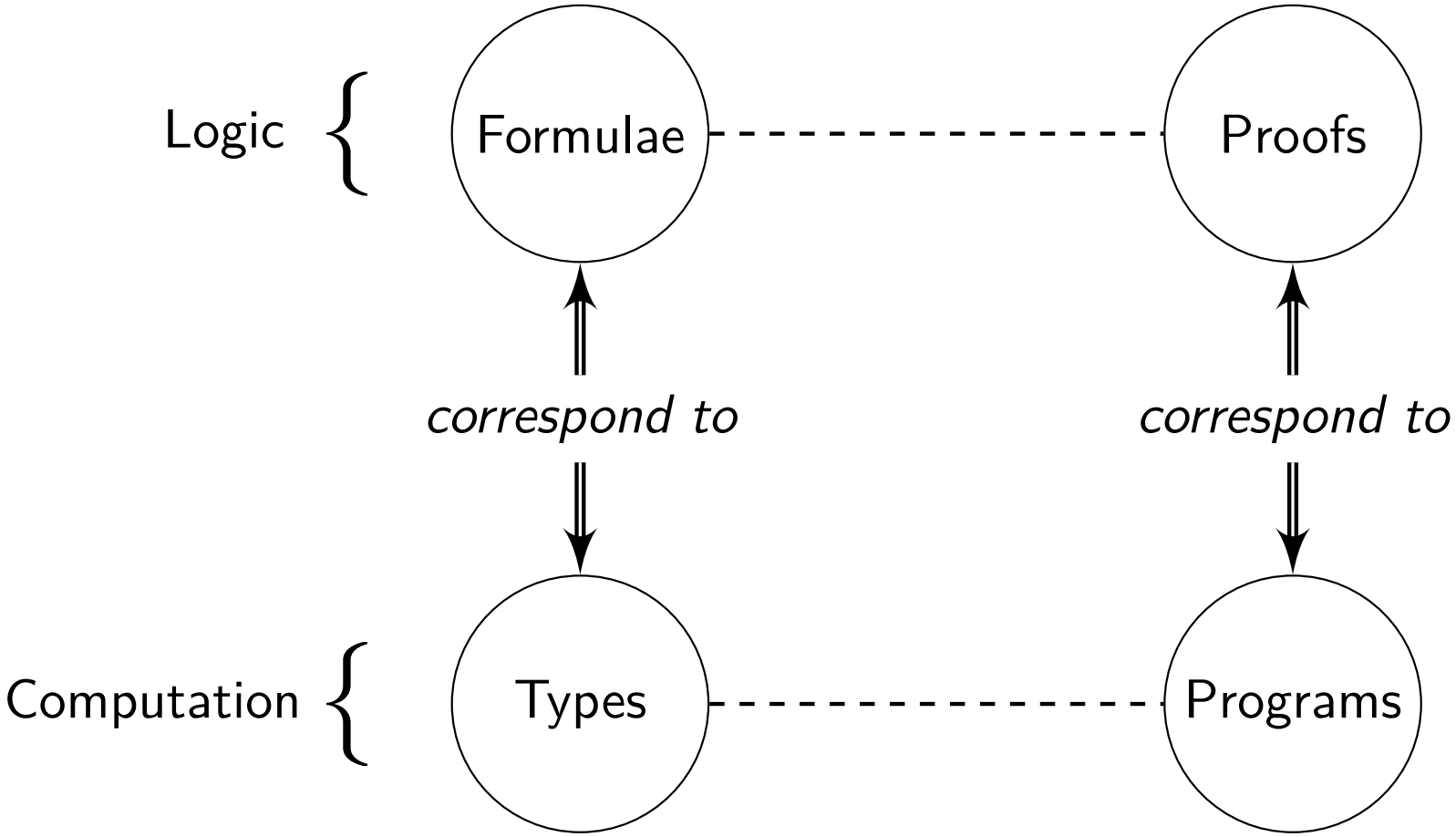
$$\Gamma_1 \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Pi_n$$

Curry–Howard correspondence



Making proofs analytic $\Longleftrightarrow$ evaluating programs

Curry–Howard correspondence



Making proofs analytic $\Longleftrightarrow$ evaluating programs

Curry–Howard correspondence II

- Introduced for Intuitionistic logic and simply typed lambda calculus
- using Natural Deduction systems

$$\dfrac{\begin{array}{c}\vdots\\ t : A \to B\end{array} \qquad \begin{array}{c}\vdots\\ u : A\end{array}}{tu : B} \qquad\qquad \dfrac{\begin{array}{c}[x : A]\\ \vdots\\ t : B\end{array}}{\lambda x\, t : A \to B}$$

- Proof Transformation $\mapsto$ β-reduction in λ-calculus
  Âă

$$\dfrac{\dfrac{\begin{array}{c}[x : A]^1\\ \vdots\\ u : B\end{array}}{\lambda x.u : A \to B}\,1 \qquad \begin{array}{c}\Gamma_1\\ \mathcal{P}\\ t : A\end{array}}{(\lambda x.u)t : B} \quad \mapsto \quad \begin{array}{c}\Gamma_1\\ \mathcal{P}\\ t : A\\ \vdots\\ u[t/x] : B\end{array}$$

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Different logics, different models of computation

- classical logic (Griffin 1990, Parigot 1997 ..)
- linear logic
- modal logics
- ...

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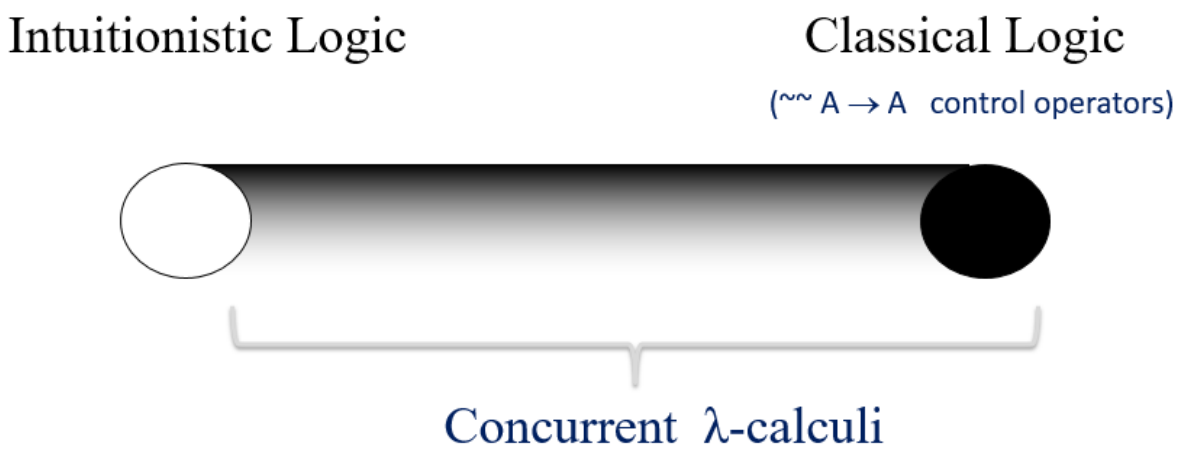


From Proof Theory to Computation

Hypersequents

$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \to B}$$

Natural Deduction

Curry–Howard for Intermediate Logics

Concurrent λ-calculi

$\lambda n\, \lambda m\, a(nm) \parallel_a \lambda p\, \lambda f\, f(pf(ax))$

From Hypersequents to Natural Deduction

Chapter II

Via ... Systems of rules

(S. Negri, JLC 2016)

Defined on labelled sequents (*e.g.*, "$xRy, \Gamma \Rightarrow \Delta, y : A$") to capture all normal modal logics formalised by Sahlqvist formulae

Sets of rules connected by:

- order constraints
- formula matching constraints

two-levels:

$$\dfrac{\dfrac{\Gamma_1^1 \Rightarrow \Delta_1^1 \;\ldots\; \Gamma_1^{n_1} \Rightarrow \Delta_1^{n_1}}{\Gamma_1 \Rightarrow \Delta_1}\,(top_1) \quad \cdots \quad \dfrac{\Gamma_k^1 \Rightarrow \Delta_k^1 \;\ldots\; \Gamma_k^{n_k} \Rightarrow \Delta_k^{n_k}}{\Gamma_k \Rightarrow \Delta_k}\,(top_k)}{\Gamma \Rightarrow \Delta \quad \ldots \quad \Gamma \Rightarrow \Delta}\,(bottom)$$

Two-systems of rules: an example

Gödel Logic $= IL + (A \to B) \vee (B \to A)$

$$\dfrac{\dfrac{\begin{array}{c} B, \Gamma_1 \Rightarrow \Delta_1 \\ \hline A, \Gamma_1 \Rightarrow \Delta_1 \end{array}\,(com_1)}{\vdots \atop \Gamma \Rightarrow \Delta} \qquad \dfrac{\dfrac{\begin{array}{c} A, \Gamma_2 \Rightarrow \Delta_2 \\ \hline B, \Gamma_2 \Rightarrow \Delta_2 \end{array}\,(com_2)}{\vdots \atop \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\,(ec)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,init.}{A \Rightarrow B}\,(com_1)}{\Rightarrow A \to B}\,(\to r)}{\Rightarrow (A \to B) \vee (B \to A)}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,init.}{B \Rightarrow A}\,(com_2)}{\Rightarrow B \to A}\,(\to r)}{\Rightarrow (A \to B) \vee (B \to A)}\,(\vee r)}{\Rightarrow (A \to B) \vee (B \to A)}\,(ec)$$

Embedding two-systems and hypersequents

(AC, Genco – TOCL 2018)

- Any hypersequent derivation can be transformed into a derivation using two systems of rules
- Any hypersequent rule can be rewritten as a two systems of rules, and viceversa

$$\frac{\mathcal{G} \mid \Gamma'_1 \Rightarrow \Delta'_1 \qquad \dots \qquad \mathcal{G} \mid \Gamma'_k \Rightarrow \Delta'_k}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n}$$

$\mathcal{S}_1, \dots, \mathcal{S}_n$ sets of sequents $\Uparrow\Downarrow$ $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_n = \{\Gamma'_i \Rightarrow \Delta'_i\}_{1 \leq i \leq k}$

$$\frac{\begin{array}{c}\mathcal{S}_1 \\ \hline \Gamma_1 \Rightarrow \Delta_1 \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \qquad \dots \qquad \begin{array}{c}\mathcal{S}_n \\ \hline \Gamma_n \Rightarrow \Delta_n \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

From Hypersequents to Natural Deduction

as a corollary of the embedding

Example: Gödel logic

$$\frac{B, \Gamma_1 \Rightarrow \Delta_1 \qquad A, \Gamma_2 \Rightarrow \Delta_2}{A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2} \xRightarrow{\textit{embedding}} \frac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}}{\vdots \atop \Gamma \Rightarrow \Delta} \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}}{\vdots \atop \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

$$\frac{\dfrac{A}{B} \atop \vdots \atop F \qquad \dfrac{B}{A} \atop \vdots \atop F}{F}$$

From Hypersequents to Natural Deduction

as a corollary of the embedding

Example: Gödel logic

$$\frac{B, \Gamma_1 \Rightarrow \Delta_1 \qquad A, \Gamma_2 \Rightarrow \Delta_2}{A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2} \xRightarrow{\textit{embedding}} \frac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}}{\vdots \atop \Gamma \Rightarrow \Delta} \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}}{\vdots \atop \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

$$\Downarrow$$

$$\frac{\dfrac{A}{B} \atop \vdots \atop F \qquad \dfrac{B}{A} \atop \vdots \atop F}{F}$$

From hypersequents to Natural Deduction

Example II: Classical Logic

$$\frac{\Gamma_1, A \Rightarrow \Delta_1}{\Gamma_1 \Rightarrow \Delta_1 \mid A \Rightarrow} \quad \xRightarrow{\textit{embedding}} \quad \frac{\dfrac{\Gamma_1, A \Rightarrow \Delta_1}{\Gamma_1 \Rightarrow \Delta_1} \qquad \dfrac{}{A \Rightarrow}}{\Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta}$$

$$\Downarrow$$

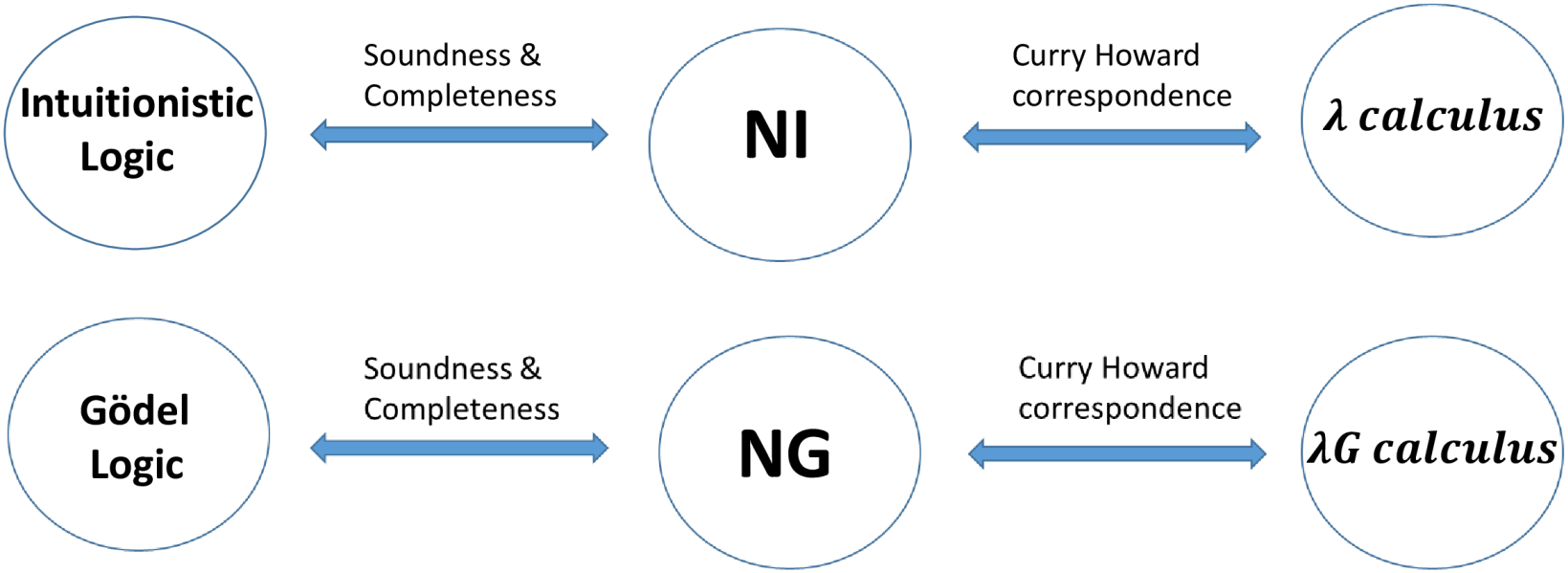
$$\frac{\dfrac{A}{\bot} \qquad [A]}{F}$$

From Hypersequents to Concurrent λ-calculi

Chapter III

A case study: Gödel logic

(Aschieri, AC and Genco – LICS 2017)



Natural deduction calculus **NG**:= **NI** + $\dfrac{\begin{array}{c}A\\ \hline B\\ \vdots\\ F\end{array} \qquad \begin{array}{c}B\\ \hline A\\ \vdots\\ F\end{array}}{F}$

The calculus λ_G

$$x^A : A \qquad \frac{\begin{array}{c}[x^A : A] \\ \vdots \\ u : B\end{array}}{\lambda x^A u : A \to B} \qquad \frac{t : A \to B \quad u : A}{tu : B}$$

$$\frac{u : A \quad t : B}{\langle u, t\rangle : A \wedge B} \qquad \frac{u : A \wedge B}{u\,\pi_0 : A} \qquad \frac{u : A \wedge B}{u\,\pi_1 : B}$$

$$\frac{\Gamma \vdash u : \bot}{\Gamma \vdash \mathrm{efq}_P(u) : P} \qquad \text{with } P \text{ atomic, } P \neq \bot.$$

$$\frac{\begin{array}{c}\dfrac{u : A}{au : B} \\ \vdots \\ w_1 : F\end{array} \qquad \begin{array}{c}\dfrac{v : B}{av : A} \\ \vdots \\ w_2 : F\end{array}}{w_1 \parallel_a w_2 : F}$$

$w_1 \parallel_a w_2$ has the role of $\nu a(\, W_1 \mid W_2)$ in π-calculus

The calculus λ_G

$$x^A : A \qquad \frac{\begin{array}{c}[x^A : A]\\ \vdots \\ u : B\end{array}}{\lambda x^A u : A \to B} \qquad \frac{t : A \to B \quad u : A}{tu : B}$$

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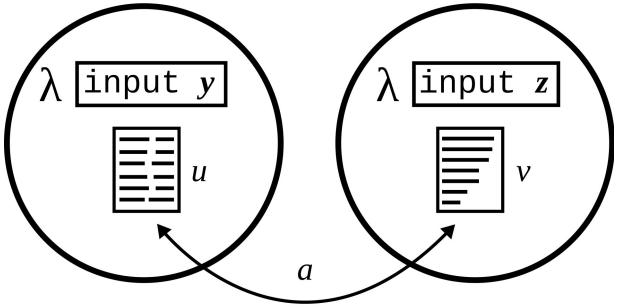
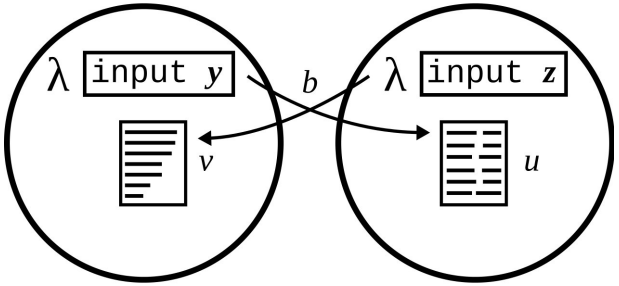
$w_1 \parallel_a w_2$ has the role of $\nu a(\, W_1 \mid W_2)$ in π-calculus

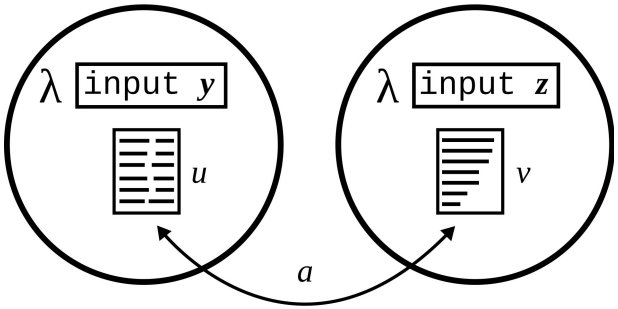
Making proofs analytic: Cross reductions & Code Mobility

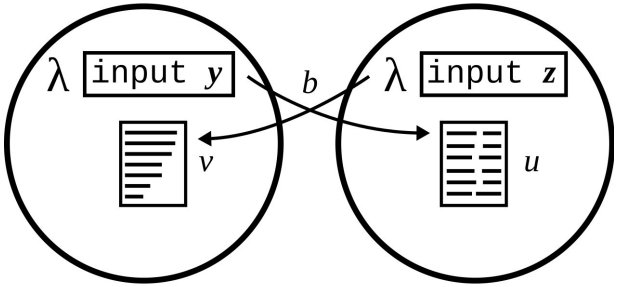




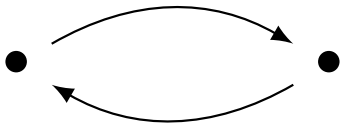
$$\dfrac{\begin{array}{c}[y]\\ u\\ \dfrac{A}{B}\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[x]\\ v\\ \dfrac{B}{A}\\ \vdots\\ C\end{array}}{C}\ a$$

Making proofs analytic: Cross reductions & Code Mobility

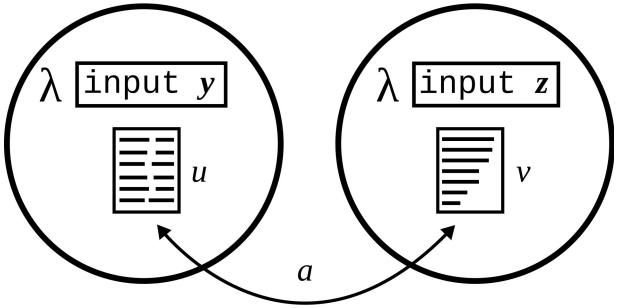


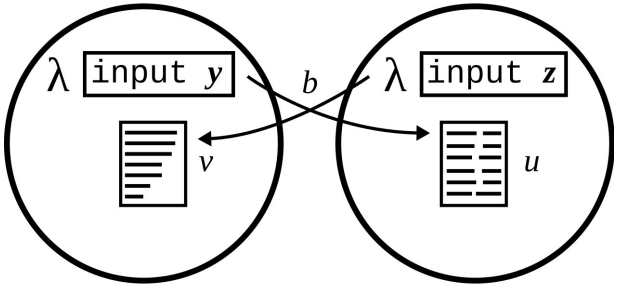


$$\dfrac{\begin{array}{c}[y]\\ u\\ \dfrac{A}{B}\\ \vdots\\ C\end{array} \quad \begin{array}{c}[x]\\ v\\ \dfrac{B}{A}\\ \vdots\\ C\end{array}}{C}\,a \qquad\qquad \dfrac{\begin{array}{c}x\\ v\\ B\\ \vdots\\ C\end{array} \quad \begin{array}{c}y\\ u\\ A\\ \vdots\\ C\end{array}}{C}$$



Making proofs analytic: Cross reductions & Code Mobility





$$\dfrac{\begin{array}{c}[y]\\ u\\ \dfrac{A}{B}\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[x]\\ v\\ \dfrac{B}{A}\\ \vdots\\ C\end{array}}{C}\,a \qquad\qquad \dfrac{\begin{array}{c}\dfrac{[x]}{y}\\ u\\ A\\ \vdots\\ C\end{array} \qquad \begin{array}{c}\dfrac{[y]}{x}\\ v\\ B\\ \vdots\\ C\end{array}}{C}\,b$$

Analytic proofs and Computation

Normalization: ideas from

- Embedding of hypersequents into systems of rules
- Hypersequent cut-elimination

1. Transform the term in parallel form

$$t_1 \parallel_{a_1} t_2 \parallel_{a_2} \cdots \parallel_{a_n} t_{n+1}$$

2. By permutations, isolate the redex $u \parallel_a v$, where a violates the subformula property with maximal complexity
3. Normalize in parallel u and v
(Simply-typed λ-calculus reductions)
4. Apply cross reductions to $u \parallel_a v$

Every proof term reduces to an **analytic proof**

Analytic proofs and Computation

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Every proof term reduces to an **analytic proof**

Computing with λ_G

- more expressive than simply typed λ calculus
- enhanced efficiency via code mobility

It's more powerful than λ-calculus

Example: Parallel disjunction, i.e.

a term O s.t. $\begin{cases} \mathsf{O}u\mathsf{T} & \mapsto^* & \mathsf{T} \\ \mathsf{O}\mathsf{T}u & \mapsto^* & \mathsf{T} \\ \mathsf{O}\mathsf{F}\mathsf{F} & \mapsto^* & \mathsf{F} \end{cases}$

Not expressible in λ-calculus (by Berry's sequentiality theorem)

In λ_G

$$\mathsf{O} := \lambda x^{\mathsf{Bool}}\, \lambda y^{\mathsf{Bool}} (\mathsf{if}\ x\ \mathsf{then}\ (\lambda z\, \lambda k\, z)\ \mathsf{else}\ (\lambda z\, \lambda k\, k))\mathsf{T}(ax)$$
$$\|_a\ (\mathsf{if}\ y\ \mathsf{then}\ (\lambda z\, \lambda k\, z)\ \mathsf{else}\ (\lambda z\, \lambda k\, k))\mathsf{T}(ay)$$

Further example: Classical logic

(F. Aschieri, AC and F. A. Genco, Submitted 2018)

Logic $IL + A \vee \neg A$ λ-calculus λ_{CL}

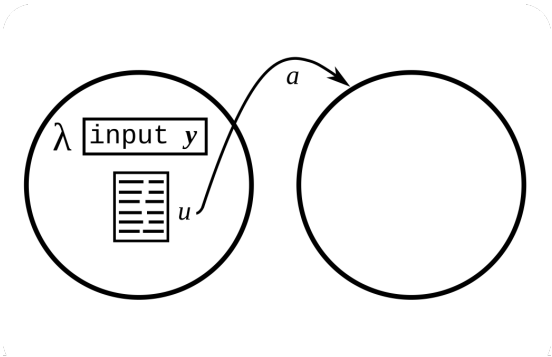
$$\dfrac{\begin{array}{c}[a^A : A] \\ \vdots \\ s : F\end{array} \qquad \begin{array}{c}\dfrac{u : A}{a^{\neg A}\, u : \bot} \\ \vdots \\ t : F\end{array}}{s \parallel_a t : F}\ (em)$$

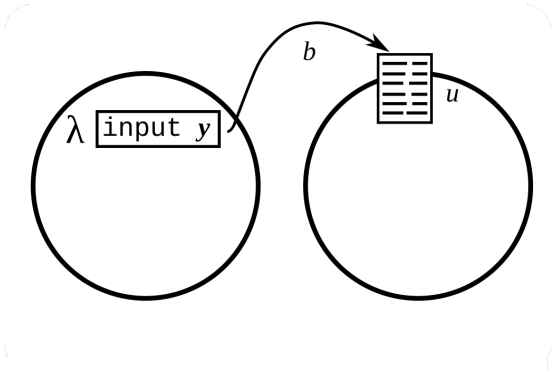
Communication schema (sending data)

$$C[\overline{a}u] \parallel_a D \to D[u/a]$$

Classical logic and code mobility

$$IL + A \vee \neg A$$





$$\dfrac{\begin{array}{c}[y]\\ u\\ \dfrac{A}{\bot}\\ \vdots\,\gamma\\ F\end{array} \qquad \begin{array}{c}[A]\\ \vdots\\ F\end{array}}{F}\;a \qquad\qquad \dfrac{\begin{array}{c}y\\ \hline\!\!\bot\\ \vdots\,\gamma\\ F\end{array} \qquad \begin{array}{c}[y]\\ u\\ A\\ \vdots\\ F\end{array}}{F}\;b$$

$$C[\overline{a}u] \parallel_a D \to (C[\overline{b}\,y] \parallel_a D) \parallel_b D[u^{y/b}/a]$$

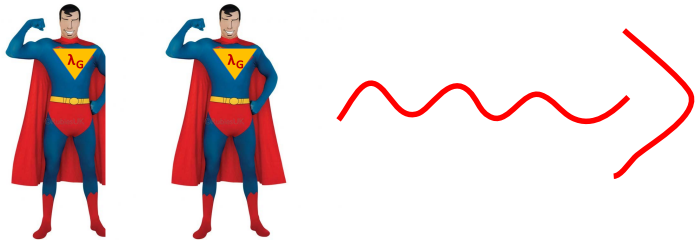
Logic-based concurrent λ-calculi

How far can we go?

CL Gödel logic

Logic-based concurrent λ-calculi

How far can we go?



Logic-based concurrent λ-calculi

For any axiom $Ax \in \mathcal{P}_3'$, let λ_{Ax} be the concurrent λ-calculus corresponding to the logic IL $+ Ax$.

Our Results (F. Aschieri, AC and F. A. Genco, Submitted 2018)

- Every proof term in λ_{Ax} reduces to a **normal** proof term satisfying the **subformula property**
- λ_{Ax} is more powerful than the λ-calculus
- λ_{Ax} allows to send open processes

Example: for $Ax := (A_1 \to A_2) \vee (A_2 \to A_3) \vee (A_3 \to A_1)$

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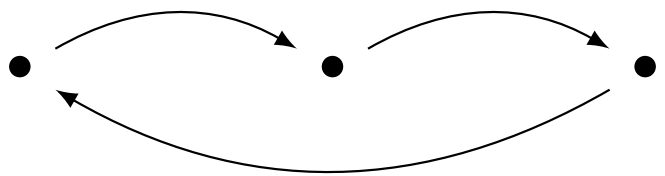
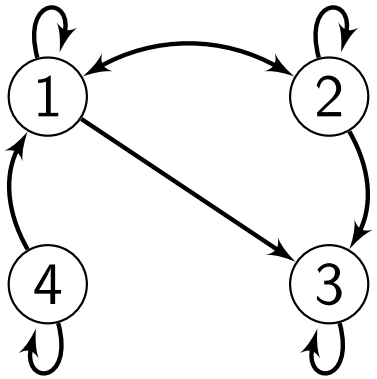
Example: for $Ax := (A_1 \to A_2) \vee (A_2 \to A_3) \vee (A_3 \to A_1)$



Process-oriented programming

- Graphs as specification for communication topologies
- Encoded as axioms of the shape $\bigvee(A_i \to \bigwedge B_j) \in \mathcal{P}_3$

Example:



encoded as

$$(A_1 \to A_1 \wedge A_2 \wedge A_4) \vee (A_2 \to A_2 \wedge A_1) \vee (A_3 \to A_3 \wedge A_1 \wedge A_2) \vee (A_4 \to A_4 \wedge \bot)$$

Open questions – a selection

- Climbing up the hierarchy

$$
\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \cdots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r] \arrow[ur] & \mathcal{N}_4 \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}
$$

- Addition of quantifiers
- Logics without contraction
- Strong normalization
- Expressive power of the introduced concurrent λ-calculi
- Their use (i) to formalize and reason about concurrent systems (ii) as bases for concurrent functional programming languages
- Relationships with other typed process models

Open questions – a selection

- Climbing up the hierarchy

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \cdots \\
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\end{tikzcd}$$

- Addition of quantifiers
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- Expressive power of the introduced concurrent λ-calculi
- Their use (i) to formalize and reason about concurrent systems (ii) as bases for concurrent functional programming languages
- Relationships with other typed process models

Permutation Reductions $(u \parallel_a v)w \mapsto uw \parallel_a vw$, if a does not occur free in w

$w(u \parallel_a v) \mapsto wu \parallel_a wv$, if a does not occur free in w
$\mathrm{efq}_P(w_1 \parallel_a w_2) \mapsto \mathrm{efq}_P(w_1) \parallel_a \mathrm{efq}_P(w_2)$
$(u \parallel_a v)\,\pi_i \mapsto u\,\pi_i \parallel_a v\,\pi_i$
$\lambda x^A\,(u \parallel_a v) \mapsto \lambda x^A\,u \parallel_a \lambda x^A\,v$
$\langle u \parallel_a v,\, w\rangle \mapsto \langle u, w\rangle \parallel_a \langle v, w\rangle$, if a does not occur free in w
$\langle w,\, u \parallel_a v\rangle \mapsto \langle w, u\rangle \parallel_a \langle w, v\rangle$, if a does not occur free in w
$(u \parallel_a v) \parallel_b w \mapsto (u \parallel_b w) \parallel_a (v \parallel_b w)$, if the communication complexity of b is greater than 0
$w \parallel_b (u \parallel_a v) \mapsto (w \parallel_b u) \parallel_a (w \parallel_b v)$, if the communication complexity of b is greater than 0

Cross Reductions

Basic Cross Reductions $\mathcal{C}[\overline{a}\ u] \parallel_a \mathcal{D} \ \mapsto\ \mathcal{D}[u/a]$
where $\overline{a} : \neg A$, $a : A$, $\mathcal{C}[a\,u], \mathcal{D}$ are normal simply typed λ-terms and $\mathcal{C}, \mathcal{D}$ simple contexts; the sequence of free variables of u is empty; a does not occur in u; b is fresh

Cross Reductions
$u \parallel_a v \mapsto u$, if a does not occur in u and $u \parallel_a v \mapsto v$, if a does not occur in v

$\mathcal{C}[\overline{a}\ u] \parallel_a \mathcal{D} \ \mapsto\ (\mathcal{C}[\overline{b}\ \langle \boldsymbol{y}\rangle] \parallel_a \mathcal{D}) \ \parallel_b\ \mathcal{D}[u^{b/\boldsymbol{y}}/a]$

where $\overline{a} : \neg A$, $a : A$, $\mathcal{C}[a\,u], \mathcal{D}$ are normal simply typed λ-terms and $\mathcal{C}, \mathcal{D}$ simple contexts; $\boldsymbol{y}$ is the (non-empty) sequence of the free variables of u which are bound in $\mathcal{C}[a\,u]$; B is the conjunction of the types of the variables in $\boldsymbol{y}$; a is rightmost; b is fresh and $\overline{b} : \neg B$, $b : B$

- **Intuitionistic Reductions**

  $(\lambda x^A\, u)t \;\mapsto\; u[t/x^A]$

  $\langle u_0\,,\, u_1\rangle\, \pi_i \;\mapsto\; u_i,$ for $i = 0, 1$

- **Permutation Reductions**

  $(u \parallel_a v)w \;\mapsto\; uw \parallel_a vw,$ if a does not occur free in w

  $w(u \parallel_a v) \;\mapsto\; wu \parallel_a wv,$ if a does not occur free in w

  $\mathrm{efq}_P(w_1 \parallel_a w_2) \;\mapsto\; \mathrm{efq}_P(w_1) \parallel_a \mathrm{efq}_P(w_2)$

  $(u \parallel_a v)\, \pi_i \;\mapsto\; u\, \pi_i \parallel_a v\, \pi_i$

  $\lambda x^A\, (u \parallel_a v) \;\mapsto\; \lambda x^A\, u \parallel_a \lambda x^A\, v$

  $\langle u \parallel_a v,\, w\rangle \;\mapsto\; \langle u, w\rangle \parallel_a \langle v, w\rangle,$ if a does not occur free in w

  $\langle w,\, u \parallel_a v\rangle \;\mapsto\; \langle w, u\rangle \parallel_a \langle w, v\rangle,$ if a does not occur free in w

  $(u \parallel_a v) \parallel_b w \;\mapsto\; (u \parallel_b w) \parallel_a (v \parallel_b w),$ if the communication complexity of b is greater than 0

  $w \parallel_b (u \parallel_a v) \;\mapsto\; (w \parallel_b u) \parallel_a (w \parallel_b v),$ if the communication complexity of b is greater than 0

- **Cross Reductions**

  $u \parallel_a v \;\mapsto\; u$ if a does not occur in u

  $u \parallel_a v \;\mapsto\; v$ if a does not occur in v

  $\mathcal{C}[a^{A\to B}\, u] \parallel_a \mathcal{D}[a^{B\to A}\, v] \quad \mapsto$

  $(\mathcal{D}[u^{b^{C\to D}\langle \boldsymbol{z}\rangle/\boldsymbol{y}}] \parallel_a \mathcal{C}[a^{A\to B}\, u]) \parallel_b (\mathcal{C}[v^{b^{D\to C}\langle \boldsymbol{y}\rangle/\boldsymbol{z}}] \parallel_a \mathcal{D}[a^{B\to A}\, v])$

  where $\mathcal{C}[a\, u], \mathcal{D}[a\, v]$ are normal simply typed λ-terms and $\mathcal{C}, \mathcal{D}$ simple contexts; $\boldsymbol{y}$ is the sequence of the free variables of u which are bound in $\mathcal{C}[a\, u]$; $\boldsymbol{z}$ is the sequence of the free variables of v which are bound in $\mathcal{D}[a\, v]$; C and D are the conjunctions of the types of the variables in $\boldsymbol{z}$ and $\boldsymbol{y}$, respectively; the displayed occurrences of a are the rightmost both in $\mathcal{C}[a\, u]$ and in $\mathcal{D}[a\, v]$; b is fresh; and the communication complexity of a is greater than 0