Constructive Canonicity of Inductive Inequalities

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Introduction and motivation. Perhaps the most important uniform methodology for proving Kripke completeness for modal logics is the notion of canonicity, which, thanks to duality, can be studied both frame-theoretically and algebraically. Frame-theoretically, canonicity can be formulated as *d-persistence*, i.e. preservation of validity from any given descriptive general frame to its underlying Kripke frame (or in other words, equivalence between validity w.r.t. admissible assignments and w.r.t. arbitrary assignments); algebraically, as preservation of validity from any given modal algebra to its canonical extension. The study of canonicity has been extended from classical normal modal logic to its many neighbouring logics, and has given rise to a rich literature. Core to this literature are two general methods for canonicity, pioneered by Sambin and Vaccaro [28] and by Ghilardi and Meloni [19].

Sambin and Vaccaro obtain canonicity for Sahlqvist formulas of classical modal logic in a frame-theoretic setting as a byproduct of correspondence. The core of their proof strategy is the observation that, whenever it exists, the first-order correspondent of a modal formula provides an equivalent rewriting of the modal formula with no occurring propositional variables, so that its validity w.r.t. admissible assignments is tantamount to its validity w.r.t. arbitrary assignments, which proves d-persistence. Sambin-Vaccaro's proof strategy, sometimes called *canonicity-via-correspondence*, has been imported to the algorithmic proof of canonicity given in [8], which achieves a uniform proof for the widest class known so far, the so-called *inductive formulas*, which significantly extends the class of Sahlqvist inequalities. In [8], the algorithm SQEMA produces the equivalent rewriting (i.e. the first-order correspondent) of the modal formula, and is shown to be successful on all inductive formulas.

Ghilardi and Meloni's work [19] shows that canonicity can be meaningfully investigated purely algebraically, in a constructive meta-theory where correspondence is not even defined, in general. Indeed, in [19], the canonical extension construction for certain bi-intuitionistic modal algebras, later applied also to general lattice and poset expansions in [18, 16], is formulated in terms of general filters and ideals, and does not depend on any form of the axiom of choice (such as the existence of 'enough' optimal filter-ideal pairs). Thus, while the constructive canonical extension need not be perfect in the sense of [24], the canonical embedding map, sending the original algebra into its canonical extension, retains the properties of denseness and compactness. Formulated category-theoretically, these

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properties make it possible for the authors of [19] to identify a class of formulas which are *constructively canonical* (i.e. the validity of which is preserved under constructive canonical extension). These inequalities are identified from the order-theoretic properties of the induced term-functions, and the preservation of their validity from an algebra to its constructive canonical extension is shown in two steps: first, from the elements of a given algebra to the closed/open elements of its canonical extension, and then from the closed/open elements to arbitrary elements. This proof strategy has the same order-theoretic underpinning of that of Jónsson [23], but is different, and was developed independently. It should be noted however that, in terms of the classes of formulas to which they apply, the results of both [19] and [23] fall within the scope of applicability of the canonicity-via-correspondence method.

Subsequent algebraic proofs of canonicity have mostly remained restricted to Sahlqvist formulas, rather than considering e.g. the wider class of inductive formulas [20]. For the latter class, the first algebraic proof of canonicity in the style of Jónsson-Ghilardi-Meloni appeared only very recently [27].

Contributions. In [10], the canonicity of inductive inequalities is proven in a constructive meta-theory, for classes of logics algebraically captured by varieties of normal and regular lattice expansions in arbitrary signatures. This result is obtained by importing Sambin-Vaccaro's proof strategy into the constructive setting of Ghilardi-Meloni, thus providing the conceptual unification of these very different perspectives which had remained a prominent open problem in the subsequent literature.

Methodology. These results are obtained using the tools and insights of *unified correspondence theory* [7], at the core of which is an *algebraic* reformulation of *correspondence* theory [12], with its ensuing algorithmic canonicity-via-correspondence argument [9]. This reformulation makes it possible to construe the computation of first-order correspondents in two phases: reduction, and translation (cf. [11] for an expanded discussion on this point). Formulas/inequalities are interpreted in the canonical extension $\mathbb{A}^\delta$ of a given algebra $\mathbb{A}$, and a calculus of rules (the *algorithm ALBA*) is applied to rewrite them into equivalent expressions with no occurring propositional variables, called pure. The pure expressions may, however, contain (non-propositional) variables known as nominals and co-nominals. If successful, achieving pure expressions completes the reduction phase. Pure expressions are already enough to implement Sambin-Vaccaro's canonicity strategy: indeed, the validity of pure expressions under assignments sending propositional variables into $\mathbb{A}$ (identified with the *admissible* assignments on $\mathbb{A}^\delta$) is tantamount to their validity w.r.t. arbitrary assignments on $\mathbb{A}^\delta$, and this establishes the canonicity of the original formula or inequality.

In a non-constructive setting, nominal and co-nominal variables range over the completely join-irreducible or meet-irreducible elements of $\mathbb{A}^\delta$. The soundness of the rewrite rules is based, in part, on the fact that the completely join-irreducible and meet-irreducible elements respectively join-generate and meet-generate the ambient algebra $\mathbb{A}^\delta$. Moreover, in this setting, completely meet- and join-irreducible elements correspond, via discrete duality, to first-order definable subsets of the dual relational semantics. Thus, the first-order frame correspondent of the original formula or inequality can be obtained by simply applying the appropriate standard translation to the pure expressions.

In a constructive setting, the situation just described is changed by the fact that we can no longer rely on the completely join-irreducible and meet-irreducible elements to respectively join-generate and meet-generate $\mathbb{A}^\delta$. However, we may fall back on the closed and open elements of $\mathbb{A}^\delta$ as complete join- and meet-generators, and adjust the interpretations of nominals and co-nominals accordingly. By doing this, the reduction phase remains sound in the non-constructive setting, and still yields canonicity. As expected, however, discrete duality is not available in general, and the possibility of translating to the relational semantics only applies modulo restriction to the setting in which discrete duality is available.

Relevance to extant research lines. These results contribute to develop a theoretical environment in which different proof techniques for canonicity and correspondence can be systematically compared and connected to each other; other similarly aimed results, which also use unified correspondence tools, appear in [14], which compares the algorithmic route to correspondence and canonicity for distributive lattice-based logics to the route via reduction to the Boolean setting, by means of Gödel-type translations, and in [27], where Sambin-Vaccaro's methodology for canonicity has been systematically connected to Jónsson's algebraic but non-constructive canonicity technique. In the specific case of canonicity, the results presented here build on techniques developed investigating the phenomenon of canonicity via pseudocorrespondence [13], and serve as base for proving constructive canonicity in the setting of mu-calculi [3]. As discussed in [6, 5], fixed points on complete general (i.e. non necessarily distributive) lattices such as those arising from common knowledge-type constructions are key for modelling processes of social interaction. Hence these canonicity results contribute to the creation of a solid mathematical backbone for applications of logic to social sciences. Moreover, the present results are of direct relevance to the problem of canonicity for possibility semantics in modal logic (cf. [22, 29]).

Recently, systematic connections have been established between unified correspondence and the computation of analytic structural rules of proper display calculi for certain axiomatic extensions of basic normal DLE-logics [21]. These connections have been also put to use in defining analytic Gentzen calculi for subintuitionistic logics [26]. In particular, a uniform argument for proving that the resulting proper display calculi are sound and conservative crucially uses the canonicity of (a certain subclass of) inductive inequalities. From these and other results, a perspective in modern structural proof theory has emerged which is based on the systematic integration of results in algebraic logic into the design of analytic calculi; the results and insights presented in this talk naturally fit also in this line of investigation.

It is interesting to observe that, while the core tools of unified correspondence have proven their versatility in settings as diverse as hybrid logics [15], many valued logics [25], monotone modal logic [17], and fixed-point logics [4, 2, 3], these tools have also acquired novel conceptual significance through the development of applications such as those discussed above [27, 21, 13] as well as more interdisciplinary applications such as [5, 6, 1]. The conceptual significance of these tools cannot be reduced exclusively to their original purpose related to correspondence theory. In this respect, the use of ALBA for constructive canonicity is yet another such instance.

References

[1] M. Bilkova, G. Greco, A. Palmigiano, A. Tzimoulis, and N. Wijnberg. The logic of resources and capabilities. Submitted. ArXiv preprint 1608.02222.

[2] W. Conradie and A. Craig. Canonicity results for mu-calculi: An algorithmic approach. *Journal of Logic and Computation*, 27(3):705–748, 2017.

[3] W. Conradie, A. Craig, A. Palmigiano, and Z. Zhao. Constructive canonicity for lattice-based fixed point logics. In *International Workshop on Logic, Language, Information, and Computation*, pages 92–109. Springer, 2017.

[4] W. Conradie, Y. Fomatati, A. Palmigiano, and S. Sourabh. Algorithmic correspondence for intuitionistic modal mu-calculus. *Theoretical Computer Science*, 564:30–62, 2015.

[5] W. Conradie, S. Frittella, A. Palmigiano, M. Piazzai, A. Tzimoulis, and N. Wijnberg. Toward an epistemic-logical theory of categorization. In *16th conference on Theoretical Aspects of Rationality and Knowledge (TARK 2017)*, volume 251 of *Electronic Proceedings in Theoretical Computer Science*, pages 170–189.

[6] W. Conradie, S. Frittella, A. Palmigiano, M. Piazzai, A. Tzimoulis, and N. Wijnberg. Categories: How I Learned to Stop Worrying and Love Two Sorts. In J. Väänänen, Å. Hirvonen, and R. de Queiroz, editors, *Logic, Language, Information, and Computation*, pages 145–164. Springer, 2016. ArXiv preprint 1604.00777.

[7] W. Conradie, S. Ghilardi, and A. Palmigiano. Unified correspondence. In A. Baltag and S. Smets, editors, *Johan van Benthem on Logic and Information Dynamics*, volume 5 of *Outstanding Contributions to Logic*, pages 933–975. Springer International Publishing, 2014.

[8] W. Conradie, V. Goranko, and D. Vakarelov. Algorithmic correspondence and completeness in modal logic. I. The core algorithm SQEMA. *Logical Methods in Computer Science*, 2006.

[9] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for distributive modal logic. *Annals of Pure and Applied Logic*, 163(3):338 – 376, 2012.

[10] W. Conradie and A. Palmigiano. Constructive canonicity of inductive inequalities. Submitted. ArXiv preprint 1603.08341.

[11] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for non-distributive logics. Submitted, arXiv:1603.08515.

[12] W. Conradie, A. Palmigiano, and S. Sourabh. Algebraic modal correspondence: Sahlqvist and beyond. *Journal of Logical and Algebraic Methods in Programming*, 91:60–84, 2017.

[13] W. Conradie, A. Palmigiano, S. Sourabh, and Z. Zhao. Canonicity and relativized canonicity via pseudo-correspondence: an application of ALBA. Submitted. ArXiv preprint 1511.04271.

[14] W. Conradie, A. Palmigiano, and Z. Zhao. Sahlqvist via translation. Submitted, arXiv:1603.08220.

[15] W. Conradie and C. Robinson. On sahlqvist theory for hybrid logics. *Journal of Logic and Computation*, 27(3):867–900, 2017.

[16] J. M. Dunn, M. Gehrke, and A. Palmigiano. Canonical extensions and relational completeness of some substructural logics. *Journal Symbolic Logic*, 70(3):713–740, 2005.

[17] S. Frittella, A. Palmigiano, and L. Santocanale. Dual characterizations for finite lattices via correspondence theory for monotone modal logic. *Journal of Logic and Computation*, 27(3):639–678, 2017.

[18] M. Gehrke and J. Harding. Bounded lattice expansions. *Journal of Algebra*, 238(1):345–371, 2001.

[19] S. Ghilardi and G. Meloni. Constructive canonicity in non-classical logics. *Annals of Pure and Applied Logic*, 86(1):1–32, 1997.

[20] V. Goranko and D. Vakarelov. Elementary canonical formulae: Extending Sahlqvist's theorem. *Annals of Pure and Applied Logic*, 141(1-2):180–217, 2006.

[21] G. Greco, M. Ma, A. Palmigiano, A. Tzimoulis, and Z. Zhao. Unified correspondence as a proof-theoretic tool. *Journal of Logic and Computation*, 2016. doi: 10.1093/logcom/exw022. ArXiv preprint 1603.08204.

[22] W. H. Holliday. Possibility frames and forcing for modal logic. 2015.

[23] B. Jónsson. On the canonicity of Sahlqvist identities. *Studia Logica*, 53:473–491, 1994.

[24] B. Jónsson and A. Tarski. Boolean algebras with operators. *American Journal of Mathematics*, 74:127–162, 1952.

[25] C. le Roux. Correspondence theory in many-valued modal logics. Master's thesis, University of Johannesburg, South Africa, 2016.

[26] M. Ma and Z. Zhao. Unified correspondence and proof theory for strict implication. *Journal of Logic and Computation*, 27(3):921–960, 2017.

[27] A. Palmigiano, S. Sourabh, and Z. Zhao. Jónsson-style canonicity for ALBA-inequalities. *Journal of Logic and Computation*, 27(3):817–865, 2017.

[28] G. Sambin and V. Vaccaro. A new proof of Sahlqvist's theorem on modal definability and completeness. *Journal of Symbolic Logic*, 54(3):992–999, 1989.

[29] Z. Zhao. Algorithmic correspondence and canonicity for possibility semantics. *arXiv preprint arXiv:1612.04957*, 2016.