Uniqueness Triples and Diamond

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LogiColloquium 2018
Udine, Italy

Thursday, 26 July 2018

This is joint work with Ari Meir Brodsky.

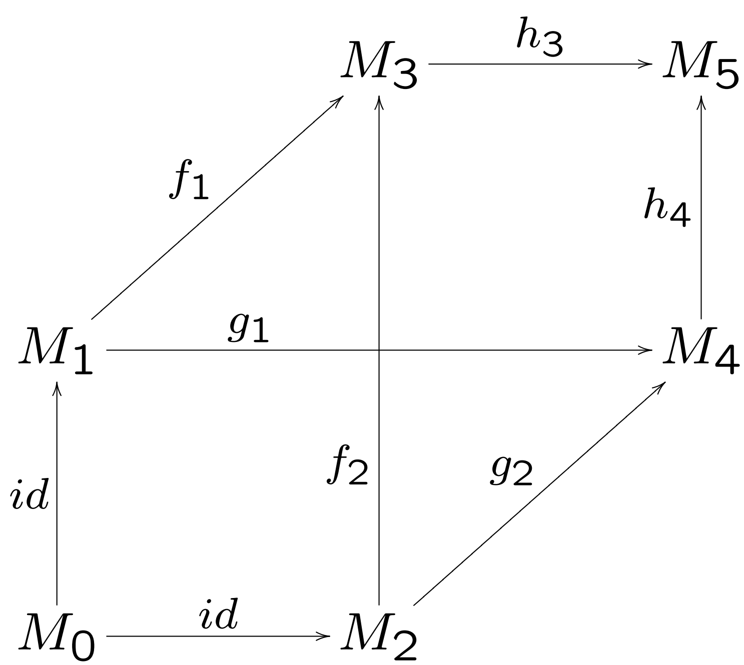
A stable (or simple) first order theory gives rise to three objects: its class K of models, the elementary submodel partial order relation $\preceq$ and the non-forking relation $\bigcup$. A pre-λ-frame is an abstraction of these three objects without specifying a theory.

Amalgamation in λ

We say that $(K, \preceq)$ satisfies the amalgamation property in λ when for every three models M_0, M_1, M_2 of cardinality λ with $M_0 \preceq M_1$ and $M_0 \preceq M_2$, we can find a model M_3 and two embeddings $f_1 : M_1 \to M_3$ and $f_2 : M_2 \to M_3$ (so $f_1[M_1] \preceq M_3$ and $f_2[M_2] \preceq M_3$) such that the following diagram commutes:

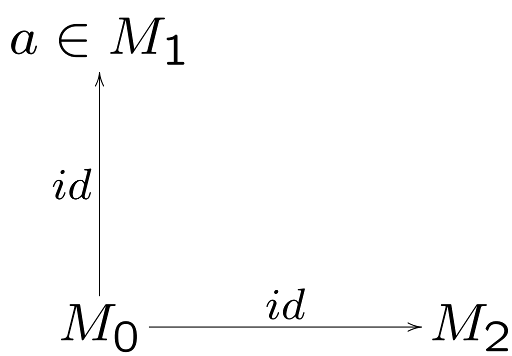
$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_3 \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle f_2} \\ M_0 & \xrightarrow{id} & M_2 \end{array}$$

The amalgamations (f_1, f_2, M_3) and (g_1, g_2, M_4) of M_1 and M_2 over M_0 are equivalent if and only if there are embeddings h_3 and h_4 so that the following diagram commutes:



We study galois types, namely orbital types.

(M_0, M_1, a) is called a uniqueness triple when $M_0 \preceq M_1$, $a \in M_1 - M_0$ and for every M_2 with $M_0 \preceq M_2$ there is a unique amalgamation of M_1 and M_2 over M_0, such that the type of a does not fork over M_0.



Theorem. [Shelah] Suppose that:

(1) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$;

(2) $\mathfrak{s}$ is a good λ-frame;

(3) $\dot{I}(\lambda^{++}, \mathbf{K}) < \mu_{\text{unif}}(\lambda^{++}, 2^\lambda) \sim 2^{\lambda^{++}}$

Then every basic triple can be extended to a uniqueness triple.

Theorem. [Main Theorem, Brodsky & Jarden, 2018] Suppose that:

(1) λ is an infinite cardinal such that $\diamondsuit(\lambda^+)$ holds;

(2) $(K, \preceq)$ is and AEC satisfying the amalgamation property in λ and in λ^+;

(3) $(K, \preceq)$ is stable in λ^+;

(4) $\mathfrak{s} = (K, \preceq_{\mathbf{K}}, S^{\mathrm{bs}}, \bigcup)$ is a pre-λ-frame satisfying transitivity, extension and continuity;

(5) A is a model of cardinality λ and p is a basic type over A.

Then there exist models C, D of cardinality λ such that $A \prec C \prec D$ and $b \in D \setminus C$ such that:

(1) (C, D, b) is a uniqueness triple;

(2) ga-tp$(b/A, D) = p$; and

(3) $\Cup(A, C, b, D)$.

$$\begin{array}{ccc} B & \longrightarrow & D \ni b \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$