Encoding Sets as Real Numbers

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PAUL HALMOS *"Naive Set Theory"*

Every mathematician agrees that every mathematician must know some set theory; the disagreement begins in trying to decide how much is some.

Sets, as they are usually conceived, have elements or members. An element of a set may be a wolf, a grape, or a pidgeon. It is important to know that a set itself may also be an element of some other set. [...] What may be surprising is not so much that sets may occur as elements, but that for mathematical purposes no other elements need ever be considered.

Definition (Hereditarily finite sets)

$\mathsf{HF} := \bigcup_{n \in \mathbb{N}} \mathsf{HF}_n$ is the collection of all *hereditarily finite* sets, where

$$\begin{cases} \mathsf{HF}_0 := \emptyset, \\ \mathsf{HF}_{n+1} := \mathscr{P}(\mathsf{HF}_n), \quad \text{for } n \in \mathbb{N}. \end{cases}$$

Definition

$$\mathbb{N}_A(x) = \Sigma_{y \in x} 2^{\mathbb{N}_A(y)}$$

Ackermann 1937

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Example

1	1	0	1	0

is the code (in base 2) of the 26th (non-empty) set, whose elements are the second, the 4th and the 5th sets in *Ackermann* order (numbering).

Ackermann 1937

Definition

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Ackermann Order

$$h_i \prec h_j \Leftrightarrow \mathbb{N}_A(h_i) = i < j = \mathbb{N}_A(h_j)$$

Ordering

$$\begin{aligned} h_i \prec h_j &\Leftrightarrow \mathbb{N}_A(h_i) = i < j = \mathbb{N}_A(h_j) \\ &\Leftrightarrow \max(h_i \setminus h_j) \prec \max(h_j \setminus h_i) \end{aligned}$$

... *which is as comparing the binary codes of i and j*

Uniqueness of (binary) positional notation

Two natural numbers are equal if and only if they have the same binary code.

Axiom of Extensionality

Two sets are equal if and only if they have the same elements.

What about *hypersets*?

Definition

$$\Omega = \{\Omega\}$$

Questions

- Extensionality?
- Can we propose a numerical encoding for hypersets?

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Extending Ackermann code: A first version

Definition (Dyadic rational numbers)

$$n/2^m \qquad n, m \in \mathbb{N}$$

$$11010, 101$$

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Mapping on dyadic rational numbers

- Find an order of (proper) *non* well-founded h.f. sets and put them on the right: let $\mathbb{Z}_A$ the result.
- Use $\mathbb{Z}_A$, that is $\mathbb{N}_A$ on the left and $\mathbb{Z}_A$ on the right.

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Example

1	1	0	1	0	,	1	0	1

is the code of $h_{26} \cup \{\hbar_1, \hbar_3\}$ with, in addition, *the first and the third non well-founded sets*.

Extending Ackermann code

Definition

$$\mathbb{R}_A(x) = \Sigma_{y \in x} 2^{-\mathbb{R}_A(y)}$$

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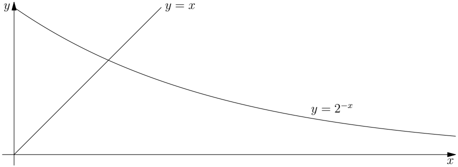

What kind of number is Ω?

- Ω is irrational.
- Ω is transcendental: (Gelfond-Schneider theorem: a^b is transcendental if a and b are algebraic, $0 \neq a \neq 1$, and b is irrational) Ω algebraic $\Rightarrow -\Omega$ algebraic $\Rightarrow$ (G.S.) $2^{-\Omega} = \Omega$ would be transcendental.
- The $\mathbb{R}_A$-code $2^{-1/\sqrt{2}}$ of $\{\emptyset\}^4$ is transcendental.
- ...

Conjectures

Conjecture 1

The function $\mathbb{R}_A$ is injective on HF^0.

Conjecture 2

The function $\mathbb{R}_A$ is injective on $\mathsf{HF}^{1/2}$.

DEFINITION (SET SYSTEMS)

A *set system* $\mathscr{S}(x_1, \ldots, x_n)$ in the set unknowns $x_1, \ldots, x_n$ is a collection of set-theoretic equations of the form:

$$\begin{cases} x_1 = \{x_{1,1}, \ldots, x_{1,m_1}\} \\ \quad\vdots \\ x_n = \{x_{n,1}, \ldots, x_{n,m_n}\}, \end{cases} \tag{1}$$

with $m_i \geq 0$ for $i \in \{1, \ldots, n\}$, and where each unknown $x_{i,u}$, for $i \in \{1, \ldots, n\}$ and $u \in \{1, \ldots, m_i\}$, occurs among the unknowns $x_1, \ldots, x_n$.

A set system $\mathscr{S}(x_1, \ldots, x_n)$ is *normal* if there exist n pairwise distinct (i.e., non bisimilar) hypersets $\hbar_1, \ldots, \hbar_n \in \mathsf{HF}^{1/2}$ such that the assignment $x_i \mapsto \hbar_i$ satisfies all the set equations of $\mathscr{S}(x_1, \ldots, x_n)$.

$\mathbb{R}_A$ is *well-given*

Any finite collection $\hbar_1, \ldots, \hbar_n$ of pairwise distinct sets in $\mathrm{HF}^{1/2}$ satisfying $\mathscr{S}(x_1, \ldots, x_n)$, univocally determines real numbers $\mathbb{R}_A(\hbar_1), \ldots, \mathbb{R}_A(\hbar_n)$ satisfying:

$$\begin{cases} \mathbb{R}_A(\hbar_1) = \sum_{k=1}^{m_1} 2^{-\mathbb{R}_A(\hbar_{1,k})} \\ \vdots \\ \mathbb{R}_A(\hbar_n) = \sum_{k=1}^{m_n} 2^{-\mathbb{R}_A(\hbar_{n,k})}. \end{cases}$$

Definition

The *multi-set approximating sequence* for (the solution of) $\mathscr{S}(x_1, \ldots, x_n)$ is

$$\left\langle \mu_i^j \mid 1 \leq i \leq n \right\rangle := \begin{cases} \langle \emptyset \mid 1 \leq i \leq n \rangle & \text{if } j = 0 \\ \left\langle \left[\mu_{i,1}^{j-1}, \ldots, \mu_{i,m_i}^{j-1} \right] \mid 1 \leq i \leq n \right\rangle & \text{if } j > 0 \,, \end{cases}$$

Definition

Code approximating sequence $\left\{\langle \mathbb{R}^{\mu}_{A}(\mu^{j}_{i}) \mid 1 \leq i \leq n\}\rangle\right\}_{j\in\mathbb{N}}$ by recursively putting:

$$\begin{cases} \mathbb{R}^{\mu}_{A}(\mu^{0}_{i}) &= 0 \\ \mathbb{R}^{\mu}_{A}(\mu^{j+1}_{i}) &= \sum_{u=1}^{m_i} 2^{-\mathbb{R}^{\mu}_{A}(\mu^{j}_{i,u})}\,. \end{cases} \tag{2}$$

We also define the corresponding *code increment sequence* $\left\{\langle \delta^{j}_{i} \mid 1 \leq i \leq n\rangle\right\}_{j\in\mathbb{N}}$ by putting, for $i \in \{1, \ldots, n\}$ and $j \in \mathbb{N}$:

$$\delta^{j}_{i} = \mathbb{R}^{\mu}_{A}(\mu^{j+1}_{i}) - \mathbb{R}^{\mu}_{A}(\mu^{j}_{i})\,. \tag{3}$$

Lemma

(i) $\mathbb{R}^{\mu}_{A}(\mu_i^{j+1}) = \delta_i^0 + \cdots + \delta_i^j$,

(ii) $\delta_i^0 = m_i$,

(iii) $\delta_i^{j+1} = \sum_{u=1}^{m_i} 2^{-\mathbb{R}^{\mu}_{A}(\mu^j_{i,u})} \cdot (2^{-\delta^j_{i,u}} - 1)$,

(iv) $\delta_i^{2j+1} \leq 0 \leq \delta_i^{2j}$,

(v) $|\delta_i^{j+1}| \leq |\delta_i^j|$, *and*

(vi) $\lim_{j\to\infty} \delta_i^j = 0$.

Theorem

For any given normal set system, the corresponding code system admits always a unique solution.

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Remark

While the value of $\mathbb{R}_A\left(\mu_i^0\right)$ is 0 for any $i \in \{1, \ldots, n\}$, the value of $\mathbb{R}_A\left(\mu_i^1\right)$—first approximation of $\mathbb{R}_A\left(\mu_i\right)$—is the cardinality of μ_i, and the subsequent approximations oscillate within the interval $[0, |\mu_i|]$. □

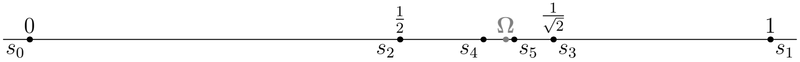
Partial results: super-singletons

Definition

The elements of the family $\mathcal{S}$ of *super-singletons*

$$\mathcal{S} \; = \; \left\{ \{\emptyset\}^{i} \mid i \in \mathbb{N} \right\},$$

are defined recursively as follows: $\{\emptyset\}^{0} = \emptyset$ and $\{\emptyset\}^{n+1} = \{\{\emptyset\}^{n}\}$.



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Proposition

The following hold:

$$0 = s_0 < \cdots s_{2i} < s_{2i+2} \cdots < \Omega < \cdots s_{2i+3} < s_{2i+1} \cdots < s_1 = 1,$$

and

$$\lim_{i\to\infty} s_{2i} = \lim_{i\to\infty} s_{2i+1} = \Omega.$$

Proposition

For all $i \in \mathbb{N}$:

1. $\mathbb{R}_A(h_i) \neq \mathbb{R}_A(h_{i+1})$;
2. $\mathbb{R}_A(h_i) \neq \mathbb{R}_A(h_{i+2})$.

Partial results: successors, density, ...

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Proposition

$\{\mathbb{R}_A(h) \mid h \in \mathsf{HF}\}$ *is dense in* $\mathbb{R}$.

Definition (The Adjunctive Hierarchy)

Let $A = \bigcup_{n \in \mathbb{N}} A_n$, where

$$\begin{aligned} A_0 &= \{\emptyset\}, \\ A_{n+1} &= \{(x \text{ with } y) \mid x, y \in A_n\} \cup \{\emptyset\}. \end{aligned}$$

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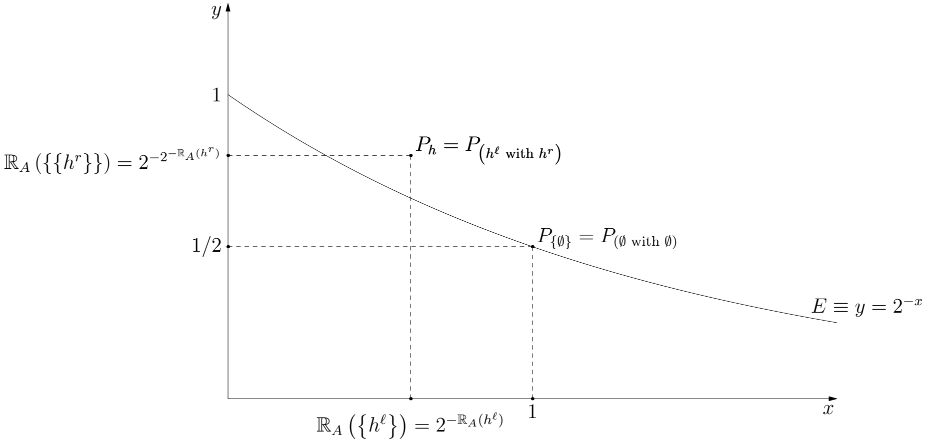
Definition (with-point)

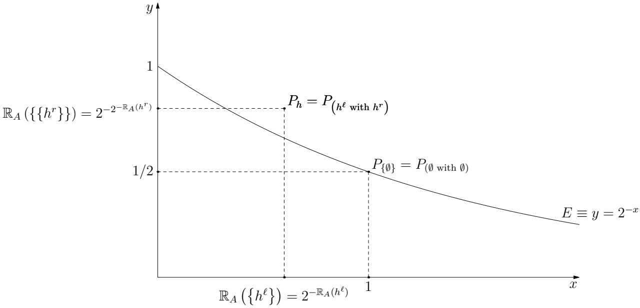
Let $h = \left(h^\ell \text{ with } h^r\right)$ be such that $h^r \notin h^\ell$ and $\mathbb{N}_A\left(h^r\right) = \max\{\mathbb{N}_A(x) \mid x \in h\}$. We denote by P_h, and call it the with-point of h, the element of $\mathbb{R}^2$

$$\left(2^{-\mathbb{R}_A\left(h^\ell\right)}, 2^{-2^{-\mathbb{R}_A\left(h^r\right)}}\right) = \left(\mathbb{R}_A(\{h^\ell\}), \mathbb{R}_A\left(\{\{\{h^r\}\}\}\right)\right).$$



y
1
$\mathbb{R}_A(\{\{h^r\}\}) = 2^{-2^{-\mathbb{R}_A(h^r)}}$
$P_h = P_{(h^\ell \text{ with } h^r)}$
1/2
$P_{\{\emptyset\}} = P_{(\emptyset \text{ with } \emptyset)}$
$E \equiv y = 2^{-x}$
1
x
$\mathbb{R}_A(\{h^\ell\}) = 2^{-\mathbb{R}_A(h^\ell)}$

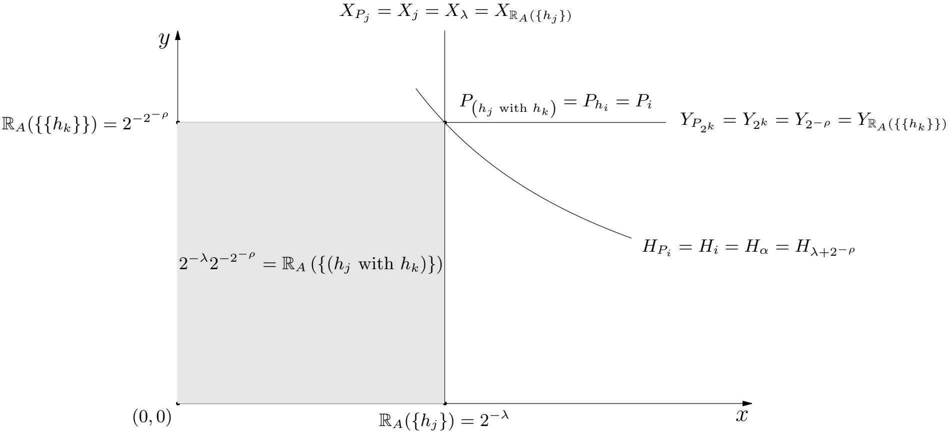
On Conjecture 1



Remark

Given $h \in \mathrm{HF}^*$, the area of the rectangle whose diagonal is the segment $[(0,0), P_h]$ is the code of $\{h\}$:

$$2^{-\mathbb{R}_A(h^\ell)} \cdot 2^{-2^{-\mathbb{R}_A(h^r)}} = 2^{-\left(\mathbb{R}_A(h^\ell)+2^{-\mathbb{R}_A(h^r)}\right)} = 2^{-\mathbb{R}_A(h)} = \mathbb{R}_A(\{h\}).$$



$X_{P_j} = X_j = X_\lambda = X_{\mathbb{R}_A(\{h_j\})}$
y
$P_{(h_j \text{ with } h_k)} = P_{h_i} = P_i$
$\mathbb{R}_A(\{\{h_k\}\}) = 2^{-2^{-\rho}}$
$Y_{P_{2^k}} = Y_{2^k} = Y_{2^{-\rho}} = Y_{\mathbb{R}_A(\{\{h_k\}\})}$
$H_{P_i} = H_i = H_\alpha = H_{\lambda+2^{-\rho}}$
$2^{-\lambda}2^{-2^{-\rho}} = \mathbb{R}_A(\{(h_j \text{ with } h_k)\})$
$(0,0)$
$\mathbb{R}_A(\{h_j\}) = 2^{-\lambda}$
x

Conjecture

For all $i, j \in \mathbb{N}^*$,

$$i \neq j \Rightarrow H_i \neq H_j.$$

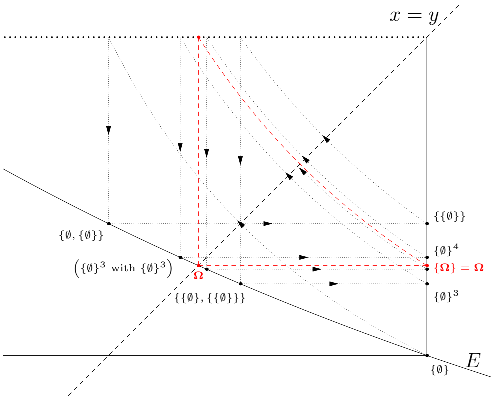


FIGURE: A geometric construction of Ω, obtained as the limit curve of the one starting from $\{\emptyset\}$ and passing through $\{\emptyset\}^n$, for $n \in \mathbb{N}^*$.

On Conjecture 2

… generalizing

Consider

$$x = \{a, x\},$$

for some fixed $a \in \mathsf{HF}^{1/2}$.
The above x has code $\mathbb{R}_A(x)$ satisfying

$$\mathbb{R}_A(x) = 2^{-\mathbb{R}_A(a)} + 2^{-\mathbb{R}_A(x)},$$

whose value we want to determine as the limit of the sequence of codes of sets

$$x^0 = \{a\},$$
$$x^{i+1} = \{a, x^i\}, \text{ for } i \in \mathbb{N}.$$

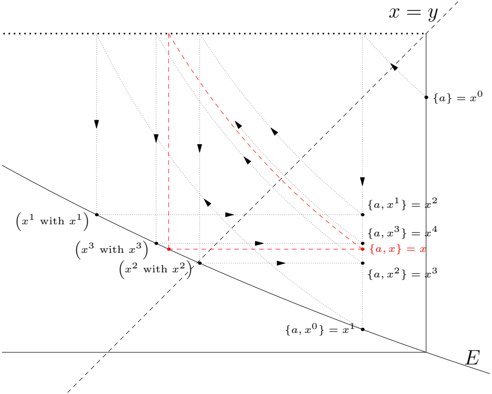


Figure: The geometric construction of the solution of $x = \{a, x\}$, obtained as the limit curve of the one starting from $x^0 = \{a\}$ and passing through $x^{n+1} = \{a, x^n\}$, for $n \in \mathbb{N}$.

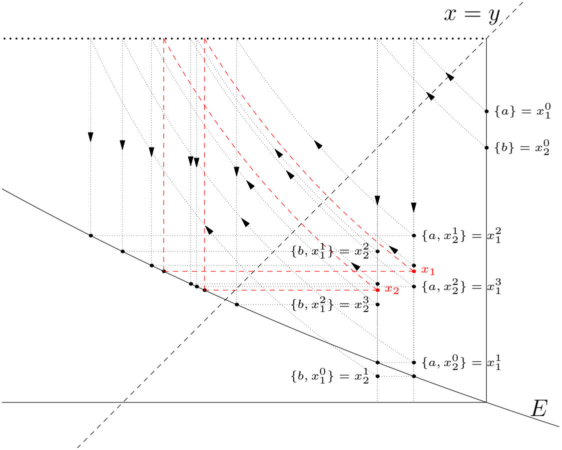


Figure: The geometric construction of the solution of $x_1 = \{a, x_2\}, x_2 = \{b, x_1\}$ obtained as the limit curve of the one starting from $x_1^0 = \{a\}, x_2^0 = \{b\}$ and passing through $x_1^{n+1} = \{a, x_2^n\}, x_2^{n+1} = \{b, x_1^n\}$, for $n \in \mathbb{N}$.

Conclusions

- Hypersets as limits of Sets
- Alternative approach to bisimulation computation
- Graphs (labelled on both nodes and edges) can be encoded/compressed by reals
- Elegant encoding