An Axiomatic Theory of Truth and Paradox

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23rd July, Logic Colloquium 2018

In this talk

What I will talk about:

- I will present an axiomatic theory of typed truth
- I will discuss its treatment of the semantic paradoxes
- I will discuss my interpretation of this treatment

What I won't talk about:

- I do not propose that the paradoxes are solved
- I do not propose I have the best treatment of the paradoxes
- I do not propose that my approach is 'revenge-immune'

. . . I am, however, happy to discuss these in the Q&A

Technical Background

We work in first order PA in the language of arithmetic $\mathscr{L}$

We expand the language with a *binary* truth predicate $Tr(x, y)$ and a rank relation $R(x, y)$

The interpretation of the truth predicate is that x is true at the level of evaluation y

The rank relation assigns formulas to levels of evaluation: ordinals coded in our language

I will be deliberately informal with coding for ease of reading, particularly with ordinals

Rank of Formulas

The rank relation $R(x, y)$ is governed by the following axioms:

1. $At_{\mathscr{L}}(x) \to R(x, \mathring{1})$
2. $R(x, y) \leftrightarrow R(\dot{\neg}x, y)$
3. $R(a \dot{\wedge} b, y) \leftrightarrow y = \max\{n, m : R(a, n) \wedge R(b, m)\}$
4. $R(a \dot{\vee} b, y) \leftrightarrow y = \max\{n, m : R(a, n) \wedge R(b, m)\}$
5. $R(\dot{\forall} a\ x(a), y) \leftrightarrow y = \sup\{y_i : R(x(i), y_i)\}$
6. $R(\dot{\exists} a\ x(a), y) \leftrightarrow y = \sup\{y_i : R(x(i), y_i)\}$
7. $R(\dot{Tr}(x, n), y) \leftrightarrow y = \begin{cases} \max\{n+1, z+1\} : & R(x, z) \wedge z > \mathring{0} \\ \mathring{0} : & R(x, z) \wedge z = \mathring{0} \end{cases}$
8. Otherwise, $R(y, \mathring{0})$.
9. $R(x, y) \to \forall n \neq y(\neg R(x, n))$

Examples

- $R(\ulcorner 0 = 0 \urcorner, \mathring{1})$
- $R(\ulcorner Tr(0 = 0, \mathring{1}) \urcorner, \mathring{2})$
- $R(\ulcorner \neg Tr(Tr(0 = 0), \mathring{1}), \mathring{2}) \urcorner, \mathring{3})$
- $R(\ulcorner \lambda_n^F \urcorner, \mathring{0})$
- $R(\ulcorner \lambda_n^F \vee 0 = 0 \urcorner, \mathring{1})$

Axiomatising Typed Truth 1

1. $At_{\mathscr{L}}(\sigma) \rightarrow (Tr(\sigma, 1) \leftrightarrow Tr^{\mathscr{L}}_{At}(\sigma))$
2. $Tr(\alpha \dot{\wedge} \beta, n) \leftrightarrow Tr(\alpha, n) \wedge Tr(\beta, n))$
3. $Tr(\alpha \dot{\vee} \beta, n) \leftrightarrow Tr(\alpha, n) \vee Tr(\beta, n))$
4. $(R(\sigma, n) \wedge n > 0) \rightarrow (Tr(\dot{\neg}\sigma, n) \leftrightarrow \neg Tr(\sigma, n))$
5. $Tr(\dot{\forall} x \sigma(x), n) \leftrightarrow \forall a Tr(\sigma(\dot{a}), n))$
6. $Tr(\dot{\exists} x \sigma(x), n) \leftrightarrow \exists a Tr(\sigma(\dot{a}), n))$

Axiomatising Typed Truth 2

1. $Tr(\dot{Tr}(\sigma, n), n+1) \leftrightarrow Tr(\sigma, n)$
2. $Tr(\sigma, n) \rightarrow \forall k{>}n\ Tr(\sigma, k)$
3. $R(\sigma, y) \rightarrow \forall k{<}y\ \neg Tr(\sigma, k)$
4. $R(\sigma, 0) \rightarrow \forall y\ \neg Tr(\sigma, y)$
5. An induction scheme for all $\mathscr{L}_{Tr}$-formulas

Structural Features of ATT

Lemma

$R(\alpha, n) \rightarrow (Tr(\alpha \dot{\rightarrow} \beta, n) \leftrightarrow (Tr(\alpha, n) \rightarrow Tr(\beta, n)))$

Theorem

$\mathbb{N}$ with Tarskian truth predicates to ϵ_0 and an appropriate rank function is a model of $\mathrm{PA} + \mathrm{ATT}$

Theorem

$\mathrm{PA} + \mathrm{ATT}$ *proves all the arithmetical consequences of* $\mathrm{PA} + \mathrm{RT}_{<\epsilon_0}$

Corollary

The arithmetic strength of $\mathrm{PA} + \mathrm{ATT}$ *is bounded below by Feferman's* $\mathrm{RA}_{<\epsilon_0}$

Alethic Features of ATT

Theorem (T-Schema)

$$\forall n[(R(\ulcorner\sigma\urcorner, n) \wedge n > 0) \rightarrow (\sigma \leftrightarrow Tr(\ulcorner\sigma\urcorner, n))]$$

Corollary (Classicality)

$$\forall n[(R(\ulcorner\varphi\urcorner, n) \wedge R(\ulcorner\psi\urcorner, n) \wedge n > 0) \rightarrow ((\varphi \leftrightarrow \psi) \leftrightarrow (Tr(\ulcorner\varphi\urcorner, n) \leftrightarrow Tr(\ulcorner\psi\urcorner, n)))]$$

Lemma (Consistency)

$$\forall\sigma\forall n\neg(Tr(\sigma, n) \wedge Tr(\neg\sigma, n))$$

Lemma (Completeness)

$$\forall\sigma\forall n\exists k[(R(\sigma, k) \wedge n \geqslant k > 0) \leftrightarrow (Tr(\sigma, n) \vee Tr(\neg\sigma, n))]$$

Rank 0 Formulas

Any $\mathscr{L}$-formula has rank 1

If we add any iterations of truth predicates to an $\mathscr{L}$-formula, then this formula has rank $n > 0$

So what sort of sentences have rank 0?

Paradoxical formulas such as: $\lambda_n^F \leftrightarrow Tr(\ulcorner \neg \lambda_n^F \urcorner, n)$

Formulas which quantify over absolutely all levels of the truth-predicate. For example:

$$\forall x[Ord(x) \rightarrow Tr(\sigma, x)]$$
$$\exists y[Ord(y) \wedge Tr(\delta, y)]$$

Paradoxes

Consider $\lambda_n^F \leftrightarrow Tr(\ulcorner \neg \lambda_n^F \urcorner, n)$

$$\mathrm{PA} + \mathrm{ATT} \vdash R(\lambda_n^F, 0) \wedge \neg \lambda_n^F$$

Consider $\lambda_n \leftrightarrow \neg Tr(\ulcorner \lambda_n \urcorner, n)$

$$\mathrm{PA} + \mathrm{ATT} \vdash R(\lambda_n, k) \wedge (k > n \vee k = 0) \wedge \lambda_n$$

Consider $\lambda \leftrightarrow \forall n[Ord(n) \rightarrow \neg Tr(\ulcorner \lambda \urcorner, n)]$

$$\mathrm{PA} + \mathrm{ATT} \vdash R(\lambda, 0) \wedge \lambda$$

Absolute Generality

One nice feature of ATT is that we can prove sentences which discuss all hierarchies of the truth predicate - e.g.: $\forall x[Ord(x) \rightarrow Tr(\sigma, x)]$

But a formula which quantifies over absolutely all hierarchies of the truth-predicate has rank 0

$$R(\forall x[Ord(x) \rightarrow Tr(\sigma, x)], k) \leftrightarrow k = sup\{y_i : R(Ord(y_i) \rightarrow Tr(\sigma, i), y_i)\}$$

Therefore k is the supremum of all ordinals, and so $k = \mathring{0}$

We can prove and disprove formulas which quantifies over absolutely all hierarchies of the truth predicate internally, but these are provably not true

Interpretation: Not ‘truth-apt’

If a formula has rank zero, then neither it nor its negation is true. It cannot be true or false.

I interpret these sentences as ‘not truth-apt’

They are syntactically valid expressions but are not the sort of sentences which are true or false

A question is one example: ‘Should I pay attention to this talk?’

Another is an imperative: ‘You won’t pay attention to this talk!’

This suggests another two kinds of examples – semantic paradoxes and unrestricted quantification

Closing Remarks

I present ATT as a modernisation of Tarski's theory of truth

The theory requires a single binary truth predicate, rather than a family of unary truth predicates

We can prove theorems about all hierarchies of the truth predicate internally (but we also prove that these are not true)

The theory provides each sentence with a rank, and distinguishes between sentences which are truth-apt, and those which are not

This provides a theory which treats the paradoxes as neither true nor false, whilst retaining classicality as much as possible

Thank you for your attention

Questions, comments, criticisms, and mistake-spottings welcome :)

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