Ramsey classes and partial orders

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Outline

Shaped Partial Order

A *shaped partial order* $\mathbf{P} = (P, <, \{\mathbf{s}^a\}_{a\in\mathcal{A}})$ is a

- partial order $(P, <)$, together with
- a finite set of unary predicates $\mathbf{s}^a$, $a \in \mathcal{A}$, satisfying
  1. $\forall p, (\mathbf{s}^1(p) \vee \mathbf{s}^2(p) \vee \ldots \vee \mathbf{s}^{|\mathcal{A}|}(p))$
  2. for each pair of distinct $a, b \in \mathcal{A}$ the statement $\forall p, \neg(\mathbf{s}^a(p) \wedge \mathbf{s}^b(p))$.

Note the following.

- We refer to $\mathbf{s}^a$, $a \in \mathcal{A}$, as *shapes*.
- The conditions (1) and (2) translate to "each point of $\mathbf{P}$ has precisely one shape."

Ramsey Class

1. Given structures A, B, we denote the set of all substructures of B isomorphic to A by $\binom{B}{A}$.
2. Given an integer k we denote the set $\{1, 2, \ldots, k\}$ by $[k]$.
3. A class $\mathcal{K}$ of structures is *Ramsey* if given any structures $A, B \in \mathcal{K}$ there exists a structure $C \in \mathcal{K}$ such that given any colouring

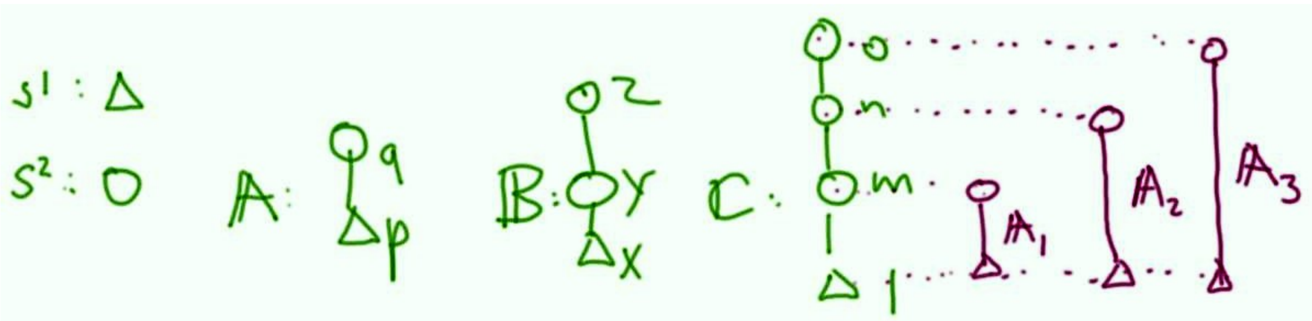
$$c : \binom{C}{A} \to [k],$$

   there exists a $B' \in \binom{C}{B}$ such that $\binom{B'}{A}$ is monochromatic.
4. We abbreviate the condition in (2) to

$$C \to (B)^A_k$$

Ramsey Example

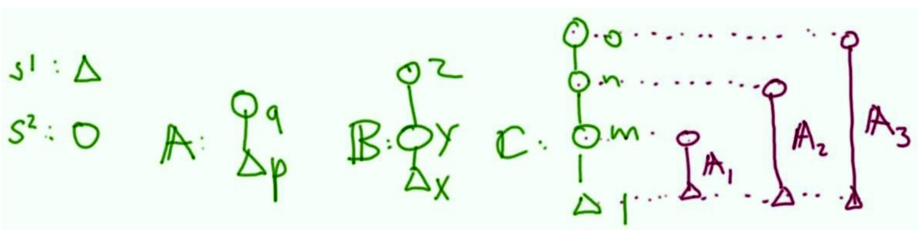
- Let $\mathcal{K}$ be a class of finite shaped total orders.
- Let $\mathbf{A}$ be a total order on points p, q with $p < q$ and $\mathbf{s}^1(p)$, $\mathbf{s}^2(q)$.
- Let $\mathbf{B}$ be a total order on points x, y, z with $x < y < z$ and $\mathbf{s}^1(x)$, $\mathbf{s}^2(y)$, $\mathbf{s}^2(z)$.
- Let $\mathbf{C}$ be a total order on points l, m, n, o with $l < m < n < o$ and $\mathbf{s}^1(l)$, $\mathbf{s}^2(m)$, $\mathbf{s}^2(n)$, $\mathbf{s}^2(o)$.
- We see that $\mathbf{C}, \mathbf{B}, \mathbf{A} \in \mathcal{K}$. We claim that $\mathbf{C} \to (\mathbf{B})_2^{\mathbf{A}}$



Ramsey Example (continued)

- The set $\binom{\mathbf{C}}{\mathbf{A}}$ contains three structures, on the subsets $\{l, m\}$, $\{l, n\}$ and $\{l, o\}$ of C.
- Since we have two colours, two of these three structures have to be of the same colour. Let those be structures on the subsets $\{l, l_1\}$ and $\{l, l_2\}$ of C, where $l_1, l_2 \in \{m, n, o\}$ and $l_1 < l_2$.
- Then we have $l < l_1 < l_2$ and $\mathbf{s}^1(l)$, $\mathbf{s}^2(l_1)$, $\mathbf{s}^2(l_2)$. So this substructure of $\mathbf{C}$ is isomorphic to $\mathbf{B}$ and has all of its substructures isomorphic to $\mathbf{A}$ of the same colour. So we indeed have

$$\mathbf{C} \to (\mathbf{B})_2^{\mathbf{A}}.$$

- 

Not Ramsey Example

- Let now $\mathcal{K}$ be a class of finite shaped antichains of chains.
- Let **A** and **B** be an antichain with two points and an antichain of two chains, each containing two points respectively, shaped as in the picture below. Let **C** be any antichain of chains.
- Order the chains of **C** in any way, suppose that it has chains

$$C_1 \prec C_2 \prec \ldots \prec C_n.$$

  Suppose that $\mathbf{A}' \in \binom{\mathbf{C}}{\mathbf{A}}$, with $\bigcirc \in C_i$ and $\triangle \in C_j$. Then **A** is a *forward* **A** if $i \prec j$ and *backward* **A** if $j \prec i$.
- Colour all forward **A**'s red and all backward **A**'s blue. Then each $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ contains a forward **A** and a backward **A**.
- So there is no **C** such that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k$. So class $\mathcal{K}$ is not Ramsey.

Classification of Ramsey classes

Nešetřil and Rödl proved that "nice" Ramsey classes are also Fraïssé.

Theorem (Nešetřil, Rödl '77)

Let $\mathcal{K}$ be a class of finite rigid structures. If $\mathcal{K}$ is a Ramsey class, hereditary, and has the joint embedding property, then $\mathcal{K}$ has the amalgamation property.

So given a classification of rigid homogeneous structures, one only needs to check whether each of the corresponding classes are Ramsey. But in many cases the homogeneous structures are not rigid, so extra steps are needed.

Topological Dynamics

Suppose that $\mathcal{H}$ is a homogeneous structure that is a totally ordered structure for $\prec$ in language $L \supseteq \{\prec\}$. Let $\mathcal{H}_0$ be a reduct of $\mathcal{H}$ to $L \setminus \{\prec\}$. If $\mathcal{H}_0$ is also homogeneous, we have the following correspondence.

$$\begin{tikzcd}
\boxed{\text{homogeneous } \mathcal{H}_0} \arrow[r, leftrightarrow, "\text{Fraïssé}"] \arrow[d, "\text{add a total order}"'] & \boxed{\text{class } \mathcal{K}_0} \arrow[d, "\text{add total orders}"] \\
\boxed{\text{ordered homogeneous } \mathcal{H}} \arrow[r, leftrightarrow, "\text{Fraïssé}"] & \boxed{\text{order class } \mathcal{K}}
\end{tikzcd}$$

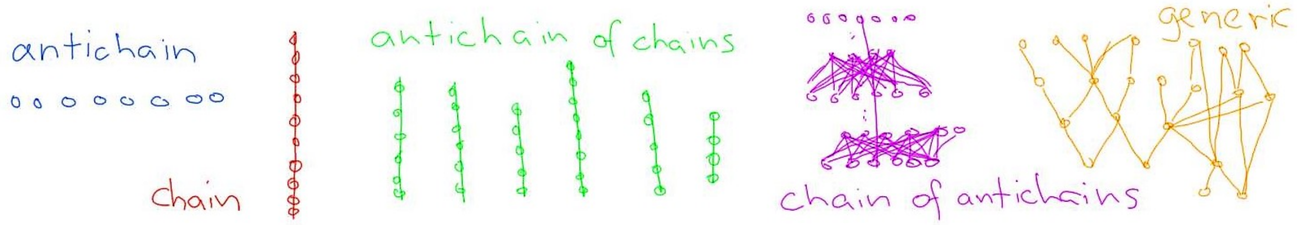
Kechris, Pestov and Todorčević show the following in paper [2].

1. If $\mathcal{K}$ is a Ramsey class then the automorphism group $Aut(\mathcal{H})$ is extremely amenable.
2. If $\mathcal{K}$ is Ramsey, *reasonable* and has the *ordering property*, then $\mathcal{K}$ provides a way to calculate the universal minimal flow of $Aut(\mathcal{H}_0)$.

Homogeneous Partial orders

Theorem (Schmerl, 79')

If $\mathcal{H}$ is a countable homogeneous partial order, then it is either an **antichain**, *a* **chain**, *an* **antichain of chains**, *a* **chain of antichains** *or a* **generic partial order**.



Context

- Sokić classified certain classes of ordered partial orders with respect to *ordering property* and *Ramsey property* and obtains the related topological dynamics results in [5].
- De Sousa and Truss classified shaped homogeneous countable partial orders $\mathcal{H}$.
    1. *Interdensely shaped components* of $\mathcal{H}$ are shaped versions of the structures on Schmerl's list.
    2. Associated with $\mathcal{H}$ is an *abstract skeleton*, a partial order together with labels for points of the poset and comparable pairs in the poset. Abstract skeletons corresponding to homogeneous structures satisfy certain conditions on the pairs and triplets of points in the skeleton.
- My work is about classes of shaped partial orders with Ramsey and ordering properties, as well as the related topological dynamics results.

Fraïssé correspondence

Definition

A countable structure $\mathcal{H}$ is *homogeneous* if any isomorphism of its finite substructures extends to an automorphism of $\mathcal{H}$.
The *age* of a homogeneous structure is a class $\mathcal{K} = \mathrm{Age}(\mathcal{H})$ of all of its finite substructures.

Theorem (Fraïssé '53)

A class $\mathcal{K}$ is an age of a homogeneous structure $\mathcal{H}$ if and only if the class $\mathcal{K}$ has hereditary property, joint embedding property, amalgamation property and contains only finitely many structures up to isomorphism, i.e., if $\mathcal{K}$ is a Fraïssé class. Further, if the class $\mathcal{K}$ satisfies the listed properties, it is an age of a homogeneous structure that is unique up to an isomorphism.

Order class

Definition

Suppose that L is a language containing a binary relation symbol $\prec$. An *order structure* $\mathbf{A}$ *for* $\prec$ is a structure $\mathbf{A}$ in language L for which $\prec^{\mathbf{A}}$ is a linear ordering. An *order class* $\mathcal{K}$ *for* $\prec$ is one for which all $\mathbf{A} \in \mathcal{K}$ are order structures for $\prec$.

We obtain classes $\mathcal{K}$ of *shaped ordered partial orders*, by extending the language $L_0 = \{<, \mathbf{s}^a\}$ of shaped partial orders to the language $L = \{<, \prec, \mathbf{s}^a\}$. Given a class $\mathcal{K}_0$ we obtain $\mathcal{K}$ by taking $(A, <, \mathbf{s}^a) \in \mathcal{K}_0$ and adding $(A, <, \prec, \mathbf{s}^a)$ to $\mathcal{K}$, for certain total orders $\prec$ on A.

Thank you!

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