What is the height of Gentzen's deduction trees?

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Der erste Widerspruchsfreiheitsbeweis für die klassische Zahlentheorie

- withdrawn in 1935 by Gentzen
- published in 1974 by Bernays

Der erste Widerspruchsfreiheitsbeweis für die klassische Zahlentheorie

- language $\mathcal{L} = \{+, \cdot, \mathrm{S}, 0, =\}$
- $\forall, \&, \neg$

- the infinitary calculus: $\dfrac{\Gamma \to F(0) \qquad \Gamma \to F(\bar{1}) \qquad \Gamma \to F(\bar{2}) \ \ldots}{\Gamma \to \forall x F(x)} \ \forall R$
  - deduction trees: cut-free
    at most countably branching
    well-founded
    sequent $\to 0 = \bar{1}$ does not have any

- first order sequent calculus for PA

Der erste Widerspruchsfreiheitsbeweis für die klassische Zahlentheorie

- Goal: Show that each sequent derivable in PA has a deduction tree.

Construction of deduction trees

- by induction on the complexity of the derivation in PA
  - axioms of PA have deduction trees of a finite height
  - construct a deduction tree for the conclusion of a derivation rule when its premises already have deduction trees
  - use the Hilfssatz when you deal with the rule of induction and rules for negation

Hilfssatz

If sequents $\Gamma \to D$ and $D, \Delta \to C$ have deduction trees T_1 and T_2, respectively, then $\Gamma, \Delta \to C$ has a deduction tree, too.

Proof of the Hilfssatz

Hilfssatz

If sequents $\Gamma \to D$ and $D, \Delta \to C$ have deduction trees T_1 and T_2, respectively, then $\Gamma, \Delta \to C$ has a deduction tree, too.

$$\dfrac{\begin{array}{c}\vdots\ T_1 \\ \Gamma \to D\end{array} \qquad \begin{array}{c}\vdots\ T_2 \\ D, \Delta \to C\end{array}}{\Gamma, \Delta \to C}$$

- Induction on the number of the logical operations in the cut formula D.
- Embedded transfinite induction on the height of T_2, the deduction tree for the second cut premise.

Formalization of the consistency proof of 1935

- TI on the height of deduction trees in the proof of the Hilfssatz = the only principle in the whole consistency proof that cannot be formalized in PA
- formalization (for example) in $I\Pi_3$ + TI on the height of deduction trees for sequents derivable in PA; TI uses at most Δ_3 induction formulas
- the notion of the deduction tree is formalizes by a Π_2 formula (in $I\Sigma_1$)

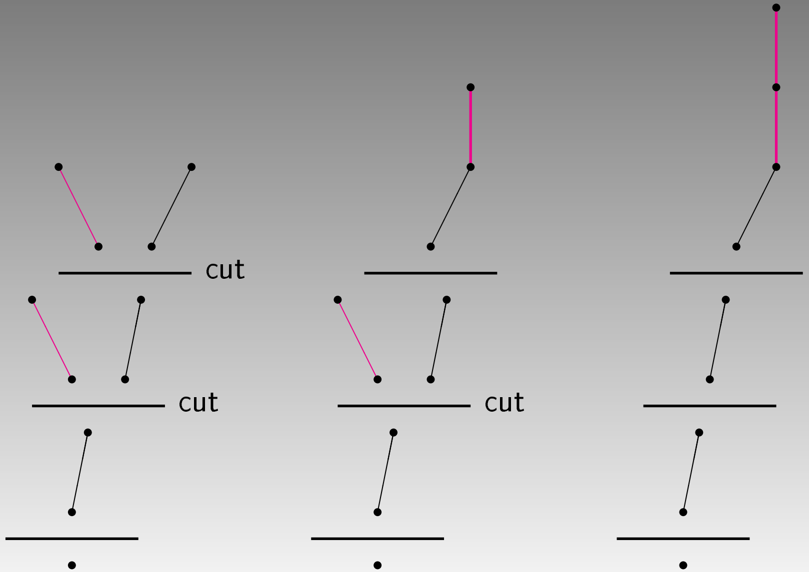
Theorem 1

Assume that sequents $\Gamma \to D$ and $D, \Delta \to C$ have deduction trees T_1 and T_2 with heights α_1 and α_2, respectively, and $|D| = n$. Then, sequent $\Gamma, \Delta \to C$ has a deduction tree whose height is at most $\Phi_{n-1}(\alpha_1 + \alpha_2)$, where $\Phi_{-1} = Id$ and Φ_n is the n-th Veblen function.

Theorem 2

Following Gentzen's procedure, we can construct for every sequent that is derivable in PA a deduction tree whose height is less than $\Phi_\omega(0)$.

Example



Thank you for your attention