The No-counterexample Interpretation in an Invertible Sequent Calculus

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What is Kreisel's no-counterexample interpretation?

- Kreisel developed an ‘interpretation’ of arithmetical formulas that can also be obtained via a double negation interpretation $+$ Gödel's Dialectica interpretation.
- It interprets arithmetical theorems of PA in a quantifier-free theory of primitive recursive functionals of finite type.

Introduction to sequent calculus

- There are standard systems of sequent calculus: LK, G0i, ...
- Consider the system G3c.
- *c* stands for classical.
- The number 3 stands for a calculus where the structural rules are admissible.
- Admissible means that we have a calculus without structural rules, but the conclusion of each structural rule can be shown to be derivable whenever the premises of the rule are derivable.

A particular property: invertibility

- A particular property that is used for proving the admissibility of the structural rules is invertibility of the logical rules.
- This means that the premises not only imply the conclusion of each rule, but the premises and the conclusion are equi-derivable.
- Not only are the logical rules invertible, but we have height-preserving invertibility. (This does not apply to the admissibility of cut.)

This leads to root-first proof search.

The NCI

- The NCI shows that there is a realizer for the Herbrand Normal Form of each theorem of PA.
- The Herbrand Normal Form is a dual version of Skolem Normal Form (Skolemization).
- When Skolemization of a Π^0_3-formula $\forall x.\exists y.\forall z.A(x, y, z)$ produces a formula $\forall x.\forall z.A(x, f_y(x), z)$ Herbrandization produces a formula $\exists y.A(c_x, y, f_z(y))$ replacing the universal quantifications.

The NCI

- If $\forall x.\exists y.\forall z.A(x, y, z)$ is derivable in G3c, then for all c_x, f_z the formula $\exists y.A(c_x, y, f_z(y))$ should be derivable.
- The NCI claims that if $\exists y.A(c_x, y, f_z(y))$ is derivable, then there is a functional $F(c_x, f_z)$, such that

$$A(c_x, F, f_z)$$

  holds.
- The name NCI comes from the equivalent statement that there should be no counterexample c_x, f_z, such that $\forall y.\neg A(c_x, y, f_z(y))$ is derivable.

The standard proof of the NCI

- Kreisel proved the NCI when he introduced it by the ϵ-substitution method (known from W. Ackermann).
- "Subsequently more perspicuous proofs of this fact via functional interpretation (combined with normalization) and cut-elimination where found." (Kohlenbach, 1997)
- The main complications arise in the case of implication.

The Schwichtenberg proof of the NCI

- Step 1 of the proof: Use an infinitary calculus with the ω-rule. Prove cut elimination.
- Step 2: Assign codes to sequents in derivations by p.r.-functions.
- Step 3: Conclude truth of an instance with a realizer in the cut-free derivation. Search is bounded by the code making the realiser functional p.r.

A proof in G3c

- A proof in G3c uses the properties of the calculus to show the condition of the NCI.
- The proof uses **a full Gödel-coding** of derivations to show that the realizer is p.r. because the derivations are finite.
- Cut elimination is implicit by having a cut-free calculus.

Extension to PA

- Let's create a calculus of G3c that is a calculus of PA.
- We retain the rules of the calculus G3c but extend it with basic arithmetic.
- We can assume that the extension is in form of rules (or axioms).
- Let's call the extended invertible calculus $G3c^*$.

We still need to add the induction schema somehow.

Prerequisites

- Herbrand's Theorem: If $\Rightarrow \exists y.D(y)$ is provable in $G3c^*$ then

$$\Rightarrow \vee_{i \in I} D(t_i)$$

  is also provable.
- Corollary: If $A \Rightarrow \exists y.D(y)$ is provable in $G3c^*$ with A purely universal, then

$$A \Rightarrow \vee_{i \in I} D(t_i)$$

  is also provable.

Remember: We need to add the induction schema.

Extension to PA

- We add the induction schema by a relativization of the derivation in $G3c^*$.
- Let $\mathcal{I}$ be an instance of the induction schema.

$$[A(0) \& (A(x) \supset A(sx))] \supset \forall x.A(x)$$

Extension to PA

- If the sequent $\Gamma \Rightarrow \Delta$ is derivable in a standard calculus of PA, then $\mathcal{I}_0, \ldots, \mathcal{I}_n, \Gamma \Rightarrow \Delta$ is derivable in $G3c^*$ for some n.
- By allowing the sequent $\Rightarrow \mathcal{I}_0 \& \ldots \& \mathcal{I}_n$ as derivable in PA we can construct a derivation.
- Note: One induction formula is sufficient by (Gentzen, 1954).

$$\dfrac{\Rightarrow \mathcal{I} \quad \begin{matrix}\Pi \\ \vdots \\ \mathcal{I}, \Gamma \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta}\ \mathit{Cut}$$

- Here the derivation Π is a derivation in $G3c^*$.

The resulting calculus is a form of PA.

Extension to PA - Form of a standard derivation

$$\dfrac{\Rightarrow \mathcal{I} \quad \begin{matrix} \Pi \\ \vdots \\ \mathcal{I}, \Gamma \Rightarrow \Delta \end{matrix}}{\Gamma \Rightarrow \Delta} \; Cut$$

Note: In this case if $\Delta \equiv \exists y.A$ and $\Gamma \equiv \emptyset$ we can assume that the induction $\mathcal{I}$ is on a purely universal formula.

$$\forall x_0 \ldots x_k.D$$

Where D is quantifier-free.

The NCI

The derivation structure in the calculus PA:

$$\dfrac{\Rightarrow \mathcal{I} \quad \dfrac{\Pi}{\mathcal{I}, \Rightarrow \forall x \exists y \forall z.A(x,y,z)}}{\Rightarrow \forall x \exists y \forall z.A(x,y,z)}\ \textit{Cut}$$

The derivation Π is in the invertible $G3c^*$.

The NCI

- By proof transformations and a final cut we get a derivation in PA that shows the equi-derivability of the Herbrand normal form.

$$\frac{\Rightarrow \mathcal{I} \quad \mathcal{I} \Rightarrow \exists y.A(x,y,f(y))}{\Rightarrow \exists y.A(x,y,f(y))}\ Cut$$

Finding the realizer

- $\mathcal{I} \Rightarrow \exists y.A(x, y, f(y))$ is derivable with an $\mathcal{I}$ induction on a purely universal formula.
- Substitute 0 for all free variables. We get a derivable sequent with only x, f as free variables.
- Apply Herbrand's theorem.
- Get a number of quantifier-free $A(x, t_i, f)$ derivable from the instances of the quantifier-free induction formulas D_i. Because D_i in the antecedent are instances of the induction formula they are true for all instances.
- There is a true formula $A(x, t_i, f)$ in the succedent. Define the functional $G(c, g) =$ the least $i < k$ such that $A(c, t_i, g)$ is true. Then define $F(x, f) = t_{G(x,f)}$.

This is our sought realizer which proves the NCI.

Finding the realizer

- Because all proof transformations are combinatorial with an upper bound on the length of the produced derivation all transformations are provably p.r. relations by a Gödel numbering.
- This implies that the realizer functional is p.r.
- Note that the bound is provided by the property of height-preserving invertibility and height-preserving admissibility of rules. (All is fine except for cut that needs an exponential growth, but there are still standard bounds).

Further reading / References:

- On the G3c: Structural Proof Theory by Negri & von Plato.
- On induction: Fusion of several complete inductions by Gerhard Gentzen in his Collected Works.
- On the NCI: The paper (Kohlenbach, 1997) or section 2.3 of (Kohlenbach, 2010).
- Good introduction on NCI: Schwichtenberg's paper in Handbook of Mathematical Logic, (ed. Barwise) 1977.