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On some universal proof system for all versions of many-valued logics

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Main notions of k-valued logic.

Let E_k be the set $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$.

Definitions of main logical functions are:

$\boldsymbol{p} \vee \boldsymbol{q} = max(p, q)$ (1) disjunction or

$\boldsymbol{p} \vee \boldsymbol{q} = [(k-1)(p+q)](mod\ k)/(k-1)$ (2) disjunction,

$\boldsymbol{p}\&q = min(p, q)$ (1) conjunction or

$\boldsymbol{p}\&q = \max(\boldsymbol{p} + q - 1, 0)$ (2) conjunction

For implication we have two following versions:

$$\boldsymbol{p} \supset \boldsymbol{q} = \begin{cases} 1, & for\ \ p \leq q \\ 1 - p + q, & for\ \ p > q \end{cases}$$ (1) Łukasiewicz's implication

or

$$\mathbf{p} \supset \boldsymbol{q} = \begin{cases} 1, & for & p \leq q \\ q, & for & p > q \end{cases}$$ (2) Gödel's implication

And for negation two versions also:

$\neg\boldsymbol{p} = 1 - p$ (1) Łukasiewicz's negation

or

$\neg\boldsymbol{p} = ((k-1)p + 1)(mod\ k)/(k-1)$ (2) cyclically permuting negation.

Sometimes we can use the notation $\overline{\boldsymbol{p}}$ instead of $\neg\boldsymbol{p}$.

For propositional variable $\boldsymbol{p}$ and $\boldsymbol{\delta}=\frac{i}{\mathrm{k}-1}(0\leq i\leq k\text{-}1)$ we define additionally **“exponent” functions:**

$\boldsymbol{p^{\delta}}$ as $(p \supset \delta)\&(\delta \supset p)$ with (1) implication (1) exponent,

$\boldsymbol{p^{\delta}}$ as p with $(k\text{-}1)-i$ (2) negations. (2) exponent.

We use the well-known notions of propositional formula, which defined as usual from propositional variables with values from $E_{k,}$ (may be also propositional constants), parentheses (,), and logical connectives $\&, \vee, \supset, \neg$. Additionaly we use two modes of exponential function p^{σ} and introduce the additional notion of formula: for every formulas A and B the expression $\boldsymbol{A^{B}}$ (for both modes) is formula also.

In the considered logics either only **1** or every of values $\frac{1}{2} \leq \frac{\boldsymbol{i}}{\mathbf{k-1}} \leq 1$ can be fixed as ***designated values***.

If we fix "**1**" (every of values $\frac{1}{2} \leq \frac{i}{\mathrm{k}-1} \leq 1$) as designated value, so a formula φ with variables $p_1, p_2, \ldots p_n$ is called ***1-k*-tautology** (***≥1/2-k-tautology***) if for every $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_k^n$ assigning δ_j ($1 \leq j \leq n$) to each p_j gives the value 1 (or some value $\frac{1}{2} \leq \frac{i}{\mathrm{k}-1} \leq 1$) of φ.

Sometimes we call ***1-k*-tautology** or ***≥1/2-k-tautology*** simply ***k*-tautology.**

Definitions of universal system for MVL and some properties of them.

Sequent type system US for all versions of MVL.

Sequent system uses the denotation of sequent $\Gamma \vdash \Delta$ where Γ (antecedent) and Δ (succedent) are finite (may be empty) sequences (or sets) of propositional formulas.

For every propositional variable p in k-valued logic $p^0, p^{1/k-1}, \ldots, p^{k-2/k-1}$ and p^1 in sense of both exponent modes are the **literals**

For every literal C and for any set of literals Γ the axiom sxeme of propositional system **US** is

$$\Gamma, C \to C.$$

For every formulas A , B, for any sets of literals Γ, each σ_1, σ_2, σ from the set E_k and for $* \in \{\&, \vee, \supset\}$ the logical rules of **US** are:

$$\vdash * \frac{\Gamma \vdash A^{\sigma_1} \text{ and } \quad \Gamma \vdash B^{\sigma_2}}{\Gamma \vdash (A * B)^{\varphi_*(A,B,\sigma_1,\sigma_2)}} \quad , \quad \vdash \exp \frac{\Gamma \vdash A^{\sigma_1} \text{ and } \quad \Gamma \vdash B^{\sigma_2}}{\Gamma \vdash (A^B)^{\varphi_{\exp}(A,B,\sigma_1,\sigma_2)}}$$

$$\vdash \neg \frac{\Gamma \vdash A^{\sigma}}{\Gamma \vdash \ (\neg A)^{\varphi_\neg(A,\sigma)}}$$

literals elimination

$$\vdash \frac{\Gamma, p^0 \vdash A, \quad \Gamma, p^{\frac{1}{k-1}} \vdash A, \quad \ldots, \quad \Gamma, p^{\frac{k-2}{k-1}} \vdash A, \quad \Gamma, p^1 \vdash A}{\Gamma \vdash A},$$

where many-valued functions

$$\varphi_*(A,B,\sigma_1,\sigma_2), \qquad \varphi_{\exp}(A,B,\sigma_1,\sigma_2), \qquad \varphi_\neg(A,\sigma),$$

must be defined individually for each version of MVL such, that

1) formulas $A^{\sigma_1} \supset (B^{\sigma_2} \supset (A * B)^{\varphi_*(A,B,\sigma_1,\sigma_2)})$,

$A^{\sigma_1} \supset (B^{\sigma_2} \supset (A^B)^{\varphi_{\exp}(A,B,\sigma_1,\sigma_2)})$ and $A^{\sigma} \supset (\neg A)^{\varphi_\neg(A,\sigma)}$

must be *k*-tautology in this version,

2) if for some σ_1, σ_2, σ the value of

$$\sigma_1 * \sigma_2 \qquad ({\sigma_1}^{\sigma_2}, \neg\sigma)$$

is one of ***designed values*** in this version of MVL , then

$(\sigma_1 * \sigma_2)^{\varphi_*(\sigma_1,\sigma_2,\sigma_1,\sigma_2)} = \sigma_1 * \sigma_2$

$(({\sigma_1}^{\sigma_2})^{\varphi_{\exp}(\sigma_1,\sigma_2,\sigma_1,\sigma_2)} = {\sigma_1}^{\sigma_2}, (\neg\sigma)^{\varphi_\neg(\sigma,\sigma)} = \neg\sigma)$.

We say that formula ***A is derived in US iff the sequent*** $\vdash$ ***A is deduced in US.***

Completeness of US

Here we give at first for the system **US** some generalization of Kalmar's proof of deducibility for two-valued tautologies in the classical propositional logic .

***Lemma* .** Let $P = \{p_1, p_2, \dots, p_n\}$ be the set of all variables of any formula A, then for every $\tilde{\delta} = (\delta_1, \delta_2, \dots, \delta_n) \in E_k^n$ the following sequent is proved in **US**.

$$p_1^{\delta_1}, p_2^{\delta_2}, \dots, p_n^{\delta_n} \vdash A^{A(\delta_1, \delta_2, \dots, \delta_n)}$$

Corollary. If A is k-tautology, then for every $\tilde{\delta} = (\delta_1, \delta_2, \dots, \delta_n) \in E_k^n$ in **US** is proved the sequent

$$p_1^{\delta_1}, p_2^{\delta_2}, \dots, p_n^{\delta_n} \vdash A.$$

Really we must use the properties 2) of the functions $\varphi_*(A, B, \sigma_1, \sigma_2)$, $\varphi_{\exp}(A, B, \sigma_1, \sigma_2)$ and $\varphi_\neg(A, \sigma)$.

Theorem Any formula is derived in **US** iff it is *k*-tautology.

Examples of US for some versions of MVL

a) For the first of constructed systems **LN_k (**Łukasiewicz's negation) with fixed "1" as designated value, use conjunction, disjunction, (1) implication, (1) negation and (1) exponent, and constants $\boldsymbol{\delta}=\frac{i}{k-1}$ (1≤i≤k-2) for using (1)exponent the functions $\varphi_*(A,B,\sigma_1,\sigma_2)$, $\varphi_{exp}(A,B,\sigma_1,\sigma_2)$, $\varphi_\neg(A,\sigma)$ are defined as follows:

$$\varphi_*(A,B,\sigma_1,\sigma_2) = \sigma_1 * \sigma_2$$

$$\varphi_{exp}(A,B,\sigma_1,\sigma_2) = {\sigma_1}^{\sigma_2}$$

$$\varphi_\neg(A,\sigma) = \neg\sigma.$$

b) For the second systems **CN₃** (cyclically permuting negation) with fixed "1" as designated value, use conjunction, disjunction,(2)implication, (2)negation and (2)exponent the functions $\boldsymbol{\varphi_{*}(A,B,\sigma_1,\sigma_2)}$, $\boldsymbol{\varphi_{exp}(A,B,\sigma_1,\sigma_2)}$, $\boldsymbol{\varphi_{\neg}(A,\sigma)}$ are defined as follows:

$$\varphi_{\supset}(A,B,\sigma_1,\sigma_2) = (\sigma_1 \supset \sigma_2)\&(\neg(A \vee \bar{A}) \vee (\bar{\bar{B}} \supset B)) \vee (\neg\left(A \vee \bar{\bar{A}}\right)\&\neg(B \vee \bar{\bar{B}})),$$

$$\varphi_{\vee}(A,B,\sigma_1,\sigma_2) = (\sigma_1 \vee \sigma_2) \vee ((A \supset \bar{A})\&\neg(\bar{B} \vee \bar{\bar{B}})) \vee (\neg\left(\bar{A} \vee \bar{\bar{A}}\right)\&(B \supset \bar{B})),$$

$$\varphi_{\&}(A,B,\sigma_1,\sigma_2) = (\sigma_1\&\sigma_2) \vee ((A\&\bar{\bar{A}}) \vee (B\&\bar{B})) \vee ((A\&\bar{A}) \vee (B\&\bar{\bar{B}})$$

$$\varphi_{\exp}(A,B,\sigma_1,\sigma_2) = {\sigma_1}^{\sigma_2} \vee (\neg({\sigma_1}^{\sigma_2})\&\neg(\neg(A^{\sigma_1}\&\bar{B}^{\sigma_2}) \vee \neg\neg(A^{\sigma_1}\&\bar{B}^{\sigma_2})))$$

$$\varphi_{\neg}(\mathrm{A},\sigma) = \neg\sigma.$$

c) For ***LN3,2* -** *Łukasiewicz's logic* with *fixed "1/2" and "1" as designated values, and with (1)* conjunction, (1) disjunction, (1) implication, (1) negation *and* (1) exponent, *and constants 0, ½ and 1 for using* (1)exponent we have

$$\varphi_*(A, B, \sigma_1, \sigma_2) = \left((A^{\sigma_1} \& B^{\sigma_2}) \& \neg(A * B)\right) \supset \neg\left((A^{\sigma_1} \& B^{\sigma_2}) \& \neg(A * B)\right)$$

$$\varphi_{exp}(A, B, \sigma_1, \sigma_2) = \left((A^{\sigma_1} \& B^{\sigma_2}) \& \neg(A^B)\right) \supset \neg\left((A^{\sigma_1} \& B^{\sigma_2}) \& \neg(A^B)\right)$$

$$\varphi_\neg(A, \sigma) = (A \& \sigma) \supset \neg(A \& \sigma)$$

The work with other version of MVL is in progress.

Thank you for attention