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On some universal proof system for all versions of many-valued logics

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Main notions of k-valued logic.

Let $E_k$ be the set $\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\}$.

Definitions of main logical functions are:

\[ p \lor q = \max(p, q) \]  \hspace{1cm} (1) disjunction or
\[ p \lor q = \left[ (k - 1)(p + q) \right] \pmod{k}/(k - 1) \]  \hspace{1cm} (2) disjunction,

\[ p \land q = \min(p, q) \]  \hspace{1cm} (1) conjunction or
\[ p \land q = \max(p + q - 1, 0) \]  \hspace{1cm} (2) conjunction
For implication we have two following versions:

\[ p \supset q = \begin{cases} 
1, & \text{for } p \leq q \\
1 - p + q, & \text{for } p > q 
\end{cases} \]  

(1) Łukasiewicz’s implication

or

\[ p \supset q = \begin{cases} 
1, & \text{for } p \leq q \\
q, & \text{for } p > q 
\end{cases} \]  

(2) Gödel’s implication

And for negation two versions also:

\[ \neg p = 1 - p \]  

(1) Łukasiewicz’s negation

or

\[ \neg p = ((k - 1)p + 1)(mod \ k)/(k - 1) \]  

(2) cyclically permuting negation.

Sometimes we can use the notation \( \overline{p} \) instead of \( \neg p \).
For propositional variable $p$ and $\delta = \frac{i}{k-1} (0 \leq i \leq k-1)$ we define additionally “exponent” functions:

$$p^\delta \quad \text{as } (p \supset \delta) \& (\delta \supset p) \text{ with (1) implication} \quad \text{(1) exponent,}$$

$$p^\delta \quad \text{as } p \text{ with } (k-1) - i \text{ (2) negations.} \quad \text{(2) exponent.}$$
We use the well-known notions of propositional formula, which defined as usual from propositional variables with values from $E_k$, (may be also propositional constants), parentheses (,), and logical connectives $\&$, $\lor$, $\Rightarrow$, $\neg$. Additionally, we use two modes of exponential function $p^\sigma$ and introduce the additional notion of formula: for every formulas $A$ and $B$ the expression $A^B$ (for both modes) is formula also.
In the considered logics either only 1 or every of values $\frac{1}{2} \leq \frac{i}{k-1} \leq 1$ can be fixed as designated values.

If we fix “1” (every of values $\frac{1}{2} \leq \frac{i}{k-1} \leq 1$) as designated value, so a formula $\varphi$ with variables $p_1, p_2, ... p_n$ is called $1$-$k$-tautology ($\geq1/2$-$k$-tautology) if for every $\tilde{\delta} = (\delta_1, \delta_2, ..., \delta_n) \in E_k^n$ assigning $\delta_j (1 \leq j \leq n)$ to each $p_j$ gives the value 1 (or some value $\frac{1}{2} \leq \frac{i}{k-1} \leq 1$) of $\varphi$.

Sometimes we call $1$-$k$-tautology or $\geq1/2$-$k$-tautology simply $k$-tautology.
Definitions of universal system for MVL and some properties of them.

**Sequent type system US for all versions of MVL.**

Sequent system uses the denotation of sequent $\Gamma \vdash \Delta$ where $\Gamma$ (antecedent) and $\Delta$ (succedent) are finite (may be empty) sequences (or sets) of propositional formulas.

For every propositional variable $p$ in $k$-valued logic $p^0, p^{1/k-1}, \ldots, p^{k-2/k-1}$ and $p^1$ in sense of both exponent modes are the **literals**
For every literal $C$ and for any set of literals $\Gamma$ the axiom schema of propositional system $\text{US}$ is

$$\Gamma, C \rightarrow C.$$  

For every formulas $A, B$, for any sets of literals $\Gamma$, each $\sigma_1, \sigma_2, \sigma$ from the set $E_k$ and for $* \in \{\&, \lor, \supset\}$ the logical rules of $\text{US}$ are:

$$\vdash * \frac{\Gamma \vdash A^{\sigma_1} \quad \Gamma \vdash B^{\sigma_2}}{\Gamma \vdash (A * B)_{\varphi(*)^{(A,B,\sigma_1,\sigma_2)}}}, \quad \vdash \exp \frac{\Gamma \vdash A^{\sigma_1} \quad \Gamma \vdash B^{\sigma_2}}{\Gamma \vdash (A^B)_{\varphi_{\exp(A,B,\sigma_1,\sigma_2)}}}$$

$$\vdash \neg \frac{\Gamma \vdash A^{\sigma}}{\Gamma \vdash (\neg A)_{\varphi_{\neg(A,\sigma)}}}$$

literals elimination

$$\vdash \frac{\Gamma, p^0 \vdash A, \quad \Gamma, p^{k-1} \vdash A, \quad \ldots, \quad \Gamma, p^{k-1} \vdash A, \quad \Gamma, p^1 \vdash A}{\Gamma \vdash A},$$
where many-valued functions

\[ \varphi_*(A, B, \sigma_1, \sigma_2), \quad \varphi_{\exp}(A, B, \sigma_1, \sigma_2), \quad \varphi_-(A, \sigma), \]

must be defined individually for each version of MVL such, that

1) formulas \( A^{\sigma_1} \supset (B^{\sigma_2} \supset (A \ast B)\varphi_*(A, B, \sigma_1, \sigma_2)) \),

\[ A^{\sigma_1} \supset (B^{\sigma_2} \supset (A^B)\varphi_{\exp}(A, B, \sigma_1, \sigma_2)) \] and \( A^{\sigma} \supset (\neg A)\varphi_-(A, \sigma) \)

must be \( k \)-tautology in this version,

2) if for some \( \sigma_1, \sigma_2, \sigma \) the value of

\[ \sigma_1 \ast \sigma_2 \quad (\sigma_1^{\sigma_2}, \neg \sigma) \]

is one of designed values in this version of MVL, then

\[ (\sigma_1 \ast \sigma_2)\varphi_*(\sigma_1, \sigma_2, \sigma_1, \sigma_2) = \sigma_1 \ast \sigma_2 \]

\[ ((\sigma_1^{\sigma_2})\varphi_{\exp}(\sigma_1, \sigma_2, \sigma_1, \sigma_2) = \sigma_1^{\sigma_2}, (\neg \sigma)\varphi_-(\sigma, \sigma) = \neg \sigma).\]
We say that formula \( A \) is derived in US iff the sequent \( \vdash A \) is deduced in US.
Completeness of US

Here we give at first for the system US some generalization of Kalmar’s proof of deducibility for two-valued tautologies in the classical propositional logic.

Lemma. Let $P = \{p_1, p_2, ..., p_n\}$ be the set of all variables of any formula $A$, then for every $\tilde{\delta} = (\delta_1, \delta_2, ..., \delta_n) \in E_k^n$ the following sequent is proved in US.

$$p_1^{\delta_1}, p_2^{\delta_2}, ..., p_n^{\delta_n} \vdash A^{A(\delta_1, \delta_2, ..., \delta_n)}$$
Corollary. If $A$ is $k$-tautology, then for every $\tilde{\delta} = (\delta_1, \delta_2, ..., \delta_n) \in E_k^n$ in US is proved the sequent

$$p_1^{\delta_1}, p_2^{\delta_2}, ..., p_n^{\delta_n} \vdash A.$$ 

Really we must use the properties 2) of the functions $\varphi_*(A, B, \sigma_1, \sigma_2)$, $\varphi_{\exp}(A, B, \sigma_1, \sigma_2)$ and $\varphi_{\neg}(A, \sigma)$. 
Theorem  Any formula is derived in US iff it is k-tautology.
Examples of US for some versions of MVL

a) For the first of constructed systems \( \text{LN}_k \) (Łukasiewicz’s negation) with fixed “1” as designated value, use conjunction, disjunction, (1) implication, (1) negation and (1) exponent, and constants \( \delta = \frac{i}{k-1} \) \((1 \leq i \leq k-2)\) for using (1)exponent the functions \( \varphi_*(A, B, \sigma_1, \sigma_2) \), \( \varphi_{\text{exp}}(A, B, \sigma_1, \sigma_2) \), \( \varphi_-(A, \sigma) \) are defined as follows:

\[
\varphi_*(A, B, \sigma_1, \sigma_2) = \sigma_1 \ast \sigma_2 \\
\varphi_{\text{exp}}(A, B, \sigma_1, \sigma_2) = \sigma_1^\sigma_2 \\
\varphi_-(A, \sigma) = \neg \sigma.
\]
b) For the second systems CN₃ (cyclically permuting negation) with fixed “1” as designated value, use conjunction, disjunction, implication, negation and exponent the functions ϕ*ₐ(B, σ₁, σ₂), ϕ*ₐ exp(A, B, σ₁, σ₂), ϕ⁻(A, σ) are defined as follows:

ϕ⁻(A, B, σ₁, σ₂) = (σ₁ ⊃ σ₂)&(¬(AVĀ)V(Ā ⊃ B))V(¬(AVĀ)&¬(B∨B)),

ϕ∨(A, B, σ₁, σ₂) = (σ₁ ∨ σ₂)V((A ⊃ A)&¬(B∨B))V(¬(ĀVĀ)&(B ⊃ B)),

ϕ∧(A, B, σ₁, σ₂) = (σ₁ & σ₂)V((A&Ā)V(B&B))V((A&Ā)V(B&B))

ϕ exp(A, B, σ₁, σ₂) = σ₁σ₂V(¬(σ₁σ₂)&¬(¬(Aσ₁&Bσ₂)V¬(Aσ₁&Bσ₂)))

ϕ⁻(A, σ) = ¬σ.
c) For $LN_{3,2}$ - Łukasiewicz's logic with fixed "1/2" and "1" as designated values, and with (1) conjunction, (1) disjunction, (1) implication, (1) negation and (1) exponent, and constants 0, $\frac{1}{2}$ and 1 for using (1) exponent we have

$$\varphi_*(A, B, \sigma_1, \sigma_2) = ((A^{\sigma_1} \& B^{\sigma_2}) \& \neg(A \ast B)) \supset \neg((A^{\sigma_1} \& B^{\sigma_2}) \& \neg(A \ast B))$$

$$\varphi_{\exp}(A, B, \sigma_1, \sigma_2) = ((A^{\sigma_1} \& B^{\sigma_2}) \& \neg(A^B)) \supset \neg((A^{\sigma_1} \& B^{\sigma_2}) \& \neg(A^B))$$

$$\varphi_-(A, \sigma) = (A \& \sigma) \supset \neg(A \& \sigma)$$

The work with other version of MVL is in progress.
Thank you for attention