Borel equivalence relations and symmetric models

Topologies for the Friedman-Stanley jumps

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Motivation

Given an equivalence relation E on a Polish space X, how does E behave *generically*?

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Definition (Kanovei-Sabok-Zapletal 2013)

An analytic equivalence relation E is **in the spectrum of the meager ideal** if there is *some* Polish topology on its domain such that $E \restriction C$ is Borel bireducible with E for any comeager set $C \subseteq X$.

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Example

The dichotomy theorems imply that E_0, E_1 and E_0^ω are in the spectrum of the meager ideal, witnessed by the natural product topologies on their domains.

Friedman-Stanley jumps

Definition

$=^+$ on $\mathbb{R}^\omega$ is defined by the complete classification

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Consider $\mathbb{R}^\omega$ with the natural product topology. If C is a comeager set, then $=^+ \restriction C$ is Borel bireducible to $=^+$.

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$=^{++}$ on $\mathbb{R}^{\omega^2}$ is defined by the complete classification

$$\langle x_{i,j} \mid i, j < \omega\rangle \mapsto \{\{x_{i,j};\ j \in \omega\}\,;\ i \in \omega\}\,.$$

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Borel equivalence relations and symmetric models

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Suppose E and F are Borel equivalence relations on X and Y respectively and $x \mapsto A_x$ and $y \mapsto B_y$ are classifications by countable structures of E and F respectively.
Assume that $f\colon X \longrightarrow Y$ is a (partial) Borel reduction from E to F. Take $x \in \operatorname{dom} f$ in some generic extension and let $A = A_x$ and $B = B_{f(x)}$. Then

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Example

Let $A \subseteq \mathbb{R}$ be a set of generic Cohen reals (a $=^+$-invariant). The "basic Cohen model" $V(A)$ is not of the form $V(r)$ for any real r. It follows that $=^+$ is not Borel reducible to $=_\mathbb{R}$ (on any comeager set).

A model of Monro (1973)

Let A^1 be the Cohen set as above.
Force over $V(A^1)$ to add a set A^2 of infinitely many generic subsets of A^1.
Consider Monro's model $V(A^1)(A^2) = V(A^2)$.

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$V(A^2) \neq V(B)$ for any set of reals B.

Since A^2 is an $=^{++}$-invariant:

Corollary

$=^{++}$ is not Borel reducible to $=^{+}$.

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F is Borel bireducible with $=^{++}$.

Corollary (of previous proof)

For any comeager set $C \subseteq \mathbb{R}^\omega \times (2^\omega)^\omega$, $F \restriction C$ is *not* Borel reducible to $=^{+}$.