Non-speed-up results for purely compositional truth predicates.

Mateusz Łełyk, Bartosz Wcisło

Institute of Mathematics, Institute of Philosophy,
University of Warsaw
Research supported by the NCN grant „Formalne teorie prawdy" („Formal Truth Theories"), no. 2014/13/B/HS1/02892

Logic Colloquium 2018, Udine, July 24, 2018

This is a report on a joint work with Ali Enayat.

Speed-up

What are speed-up questions about?

Suppose we are given two theories $\mathrm{Th}_1 \subseteq \mathrm{Th}_2$ (possibly in different languages) and we already know that Th_2 conservatively extends Th_1.

What are speed-up questions about?

Suppose we are given two theories $\mathrm{Th}_1 \subseteq \mathrm{Th}_2$ (possibly in different languages) and we already know that Th_2 conservatively extends Th_1. One of the most natural questions to ask in the next step is whether Th_2 proves theorems of Th_1 in a more efficient way.

What are speed-up questions about?

Suppose we are given two theories $\mathrm{Th}_1 \subseteq \mathrm{Th}_2$ (possibly in different languages) and we already know that Th_2 conservatively extends Th_1. One of the most natural questions to ask in the next step is whether Th_2 proves theorems of Th_1 in a more efficient way. We ask whether with the help of Th_2's axioms we can prove theorems of Th_1 significantly faster.

Lengths of proofs

We work with theories formulated in languages extending arithmetical signature $\{0, S, +, \times\}$.

Lengths of proofs

We work with theories formulated in languages extending arithmetical signature $\{0, S, +, \times\}$.

Definition

The length of the proof is the number of symbols in (the binary code of) the proof.

Lengths of proofs

We work with theories formulated in languages extending arithmetical signature $\{0, S, +, \times\}$.

Definition

The length of the proof is the number of symbols in (the binary code of) the proof.

Lengths of proofs

We work with theories formulated in languages extending arithmetical signature $\{0, S, +, \times\}$.

Definition

The length of the proof is the number of symbols in (the binary code of) the proof.

Let us define

$$\| \phi \|_T = \begin{cases} \text{the length of the shortest proof of } \phi, \text{ if } T \vdash \phi \\ \infty \text{ otherwise} \end{cases}$$

Remark

The length of the proof is **not** the number of steps (proof lines) in it. The size of formulae matters.

Speed-up

Definition (Speed-up)

Let Th_1 and Th_2 be two theories.

Speed-up

Definition (Speed-up)

Let Th_1 and Th_2 be two theories.

Speed-up

Definition (Speed-up)

Let Th_1 and Th_2 be two theories. We shall say that Th_2 has a *superpolynomial speed-up* if there exists an infinite sequence of formulae

Speed-up

Definition (Speed-up)

Let Th_1 and Th_2 be two theories. We shall say that Th_2 has a *superpolynomial speed-up* if there exists an infinite sequence of formulae

$$\phi_0, \phi_1, \ldots,$$

Speed-up

Definition (Speed-up)

Let Th_1 and Th_2 be two theories. We shall say that Th_2 has a *superpolynomial speed-up* if there exists an infinite sequence of formulae

$$\phi_0, \phi_1, \ldots,$$

provable in both Th_1 and Th_2 such that for every polynomial p for sufficiently large n

$$\| \phi_n \|_{\mathrm{Th}_1} > p(\| \phi_n \|_{\mathrm{Th}_2}).$$

Speed-up

Definition (Speed-up)

Let Th_1 and Th_2 be two theories. We shall say that Th_2 has a *superpolynomial speed-up* if there exists an infinite sequence of formulae

$$\phi_0, \phi_1, \ldots,$$

provable in both Th_1 and Th_2 such that for every polynomial p for sufficiently large n

$$\| \phi_n \|_{\mathrm{Th}_1} > p(\| \phi_n \|_{\mathrm{Th}_2}).$$

In a similar way, we can define superexponential speed-up, super-computable speed-up etc.

Some very easy examples. Let PAT^- be a theory extending PA with a unary predicate T and no extralogical axioms for this new predicate T.

Some very easy examples. Let PAT^- be a theory extending PA with a unary predicate T and no extralogical axioms for this new predicate T. It is clear that PAT^- has at most linear speed-up over PA.

Some very easy examples. Let PAT^{-} be a theory extending PA with a unary predicate T and no extralogical axioms for this new predicate T. It is clear that PAT^{-} has at most linear speed-up over PA. Namely, take any proof

$$\phi_1, \ldots, \phi_n$$

Some very easy examples. Let PAT^- be a theory extending PA with a unary predicate T and no extralogical axioms for this new predicate T. It is clear that PAT^- has at most linear speed-up over PA. Namely, take any proof

$$\phi_1, \ldots, \phi_n$$

with conclusion in L_a.

Some very easy examples. Let PAT^- be a theory extending PA with a unary predicate T and no extralogical axioms for this new predicate T. It is clear that PAT^- has at most linear speed-up over PA. Namely, take any proof

$$\phi_1, \ldots, \phi_n$$

with conclusion in L_a. Then

$$\phi_1[x = x/T(x)], \ldots, \phi_n[x = x/T(x)]$$

Some very easy examples. Let PAT^{-} be a theory extending PA with a unary predicate T and no extralogical axioms for this new predicate T. It is clear that PAT^{-} has at most linear speed-up over PA. Namely, take any proof

$$\phi_1, \ldots, \phi_n$$

with conclusion in L_a. Then

$$\phi_1[x = x/T(x)], \ldots, \phi_n[x = x/T(x)]$$

is a proof of the same sentence (since ϕ_n does not contain the predicate T).

Some very easy examples. Let PAT^- be a theory extending PA with a unary predicate T and no extralogical axioms for this new predicate T. It is clear that PAT^- has at most linear speed-up over PA. Namely, take any proof

$$\phi_1, \ldots, \phi_n$$

with conclusion in L_a. Then

$$\phi_1[x = x/T(x)], \ldots, \phi_n[x = x/T(x)]$$

is a proof of the same sentence (since ϕ_n does not contain the predicate T). It is at most 3 times as big as the original proof.

No significant speed-up

Theorem (Hajek (1993), Avigad (1996))

WKL_0 *has at most polynomial speed-up over* $\mathsf{I}\Sigma_1$.

A very concise argument was given by Wong (2016). This can be proved by showing that there exists an ω-interpretation of WKL_0 in $\mathsf{I}\Sigma_1$ (WKL_0 is finitely axiomatizable).

Some classical examples of conservative extensions with superexponential speed-up are as follows:

Some classical examples of conservative extensions with superexponential speed-up are as follows:

1. GB over ZFC.

Some classical examples of conservative extensions with superexponential speed-up are as follows:

1. GB over ZFC.
2. ACA_0 over PA.

Some classical examples of conservative extensions with superexponential speed-up are as follows:

1. GB over ZFC.
2. ACA_0 over PA.

Some classical examples of conservative extensions with superexponential speed-up are as follows:

1. GB over ZFC.
2. ACA_0 over PA.

We have no time to go into this in detail. One can prove the following general theorem:

Theorem (Pudlák, Fischer)

Let Th *be a finite extension of* PA. *Suppose that there exists a formula* $I(x)$ *defining a cut such that*

$$\mathrm{PA} \vdash I(x) \rightarrow \text{ ''there is no proof of } 0 = 1 \text{ of length less than } x\text{.''}$$

Then Th *has a superexponential speed-up.*

Compositional Truth

Axiomatic Theories of Truth

In the field of Axiomatic Theories of Truth we deal with theories built in the following way:

1. we fix a theory Th, which is sufficiently strong to formalize syntax;

Axiomatic Theories of Truth

In the field of Axiomatic Theories of Truth we deal with theories built in the following way:

1. we fix a theory Th, which is sufficiently strong to formalize syntax;
2. we extend its language with a new unary predicate $T(x)$ (denote $\mathcal{L}_T$) and add to Th some axioms for it obtaining a theory capable of proving
$$T(\underline{\phi}) \equiv \phi$$
for every $\phi \in \mathcal{L}_{\mathsf{Th}}$.

Compositional Truth

Let $\mathrm{Th} \supseteq I\Delta_0 + \exp$.

Definition

$\mathrm{CT}^-(\mathrm{Th})$ is the theory extending Th with the following sentences:

Compositional Truth

Let $\mathrm{Th} \supseteq I\Delta_0 + \exp$.

Definition

$\mathrm{CT}^-(\mathrm{Th})$ is the theory extending Th with the following sentences:

CT1 $\forall s, t \quad T(s = t) \equiv \mathrm{val}(s) = \mathrm{val}(t)$.

Compositional Truth

Let $\mathrm{Th} \supseteq I\Delta_0 + \exp$.

Definition

$\mathrm{CT}^-(\mathrm{Th})$ is the theory extending Th with the following sentences:

CT1 $\forall s, t \quad T(s = t) \equiv \mathrm{val}(s) = \mathrm{val}(t)$.

CT2 $\forall \phi, \psi \quad T(\phi \vee \psi) \equiv T\phi \vee T\psi$.

Compositional Truth

Let Th $\supseteq I\Delta_0 + \exp$.

Definition

$\mathrm{CT}^-(\mathrm{Th})$ is the theory extending Th with the following sentences:

CT1 $\forall s, t \;\; T(s = t) \equiv \mathrm{val}(s) = \mathrm{val}(t)$.

CT2 $\forall \phi, \psi \;\; T(\phi \vee \psi) \equiv T\phi \vee T\psi$.

CT3 $\forall \phi \;\; T(\neg\phi) \equiv \neg T\phi$.

Compositional Truth

Let $\mathrm{Th} \supseteq I\Delta_0 + \exp$.

Definition

$CT^-(\mathrm{Th})$ is the theory extending Th with the following sentences:

CT1 $\forall s, t \quad T(s = t) \equiv \mathrm{val}(s) = \mathrm{val}(t)$.

CT2 $\forall \phi, \psi \quad T(\phi \vee \psi) \equiv T\phi \vee T\psi$.

CT3 $\forall \phi \quad T(\neg \phi) \equiv \neg T\phi$.

CT4 $\forall y \forall \phi(y) \quad T(\exists y \phi) \equiv \exists x T\phi[\underline{x}/y]$.

Compositional Truth

Let Th $\supseteq I\Delta_0 + \exp$.

Definition

$CT^-(\mathrm{Th})$ is the theory extending Th with the following sentences:

CT1 $\forall s, t \quad T(s = t) \equiv \mathrm{val}(s) = \mathrm{val}(t)$.

CT2 $\forall \phi, \psi \quad T(\phi \vee \psi) \equiv T\phi \vee T\psi$.

CT3 $\forall \phi \quad T(\neg \phi) \equiv \neg T\phi$.

CT4 $\forall y \forall \phi(y) \quad T(\exists y \phi) \equiv \exists x T\phi[\underline{x}/y]$.

CT5 $\forall \phi(x) \forall s, t \quad (\mathrm{val}(s) = \mathrm{val}(t) \rightarrow T\phi[s/x] \equiv T\phi[t/x])$

Compositional Truth

Let Th $\supseteq I\Delta_0 + \exp$.

Definition

$CT^-(Th)$ is the theory extending Th with the following sentences:

CT1 $\forall s, t \quad T(s = t) \equiv val(s) = val(t)$.

CT2 $\forall \phi, \psi \quad T(\phi \vee \psi) \equiv T\phi \vee T\psi$.

CT3 $\forall \phi \quad T(\neg\phi) \equiv \neg T\phi$.

CT4 $\forall y \forall \phi(y) \quad T(\exists y \phi) \equiv \exists x T\phi[\underline{x}/y]$.

CT5 $\forall \phi(x) \forall s, t \quad (val(s) = val(t) \rightarrow T\phi[s/x] \equiv T\phi[t/x])$

Remark

Note that in $CT^-(PA)$ we do not have induction axioms for formulae with the truth predicate.

Comparing the axioms for the truth predicate

When investigating axiomatic theories of truth we are interested in determining which axioms are responsible for such metalogical properties as:

1. syntactical non-conservativity. We are trying to characterize the axioms for $T(x)$ which enable us to prove more sentences in the language of the base theory than the base theory itself.

Comparing the axioms for the truth predicate

When investigating axiomatic theories of truth we are interested in determining which axioms are responsible for such metalogical properties as:

1. syntactical non-conservativity. We are trying to characterize the axioms for $T(x)$ which enable us to prove more sentences in the language of the base theory than the base theory itself.
2. semantical non-conservativity. For a given theory of truth Th we are trying to characterize the class of models of PA that admit an **expansion** to a model of Th.

Comparing the axioms for the truth predicate

When investigating axiomatic theories of truth we are interested in determining which axioms are responsible for such metalogical properties as:

1. syntactical non-conservativity. We are trying to characterize the axioms for $T(x)$ which enable us to prove more sentences in the language of the base theory than the base theory itself.
2. semantical non-conservativity. For a given theory of truth Th we are trying to characterize the class of models of PA that admit an **expansion** to a model of Th.
3. speed-up.

Conservativity of CT^-

Theorem (Krajewski-Kotlarski-Lachlan (1981), Enayat-Visser (2015), Leigh (2015))

$\mathrm{CT}^-(\mathrm{PA})$ *is conservative over* PA.

Conservativity of CT^-

Theorem (Krajewski-Kotlarski-Lachlan (1981), Enayat-Visser (2015), Leigh (2015))

$CT^-(PA)$ *is conservative over* PA.

Conservativity of CT^-

Theorem (Krajewski-Kotlarski-Lachlan (1981), Enayat-Visser (2015), Leigh (2015))

$\mathrm{CT}^-(\mathrm{PA})$ *is conservative over* PA. *In fact for any* $\mathrm{Th} \supseteq I\Delta_0 + \exp$, $\mathrm{CT}^-(\mathrm{Th})$ *is conservative over* Th.

Conservativity of CT^-

Theorem (Krajewski-Kotlarski-Lachlan (1981), Enayat-Visser (2015), Leigh (2015))

$\mathrm{CT}^-(\mathrm{PA})$ *is conservative over* PA. *In fact for any* $\mathrm{Th} \supseteq I\Delta_0 + \exp$, $\mathrm{CT}^-(\mathrm{Th})$ *is conservative over* Th.

However, $\mathrm{CT}^-(\mathrm{PA})$ is quite strong from the semantical point of view.

Theorem (Lachlan, see Kaye (1991))

If $\mathcal{M} \models \mathrm{PA}$ *expands to a model of* $\mathrm{CT}^-(\mathrm{PA})$, *then* $\mathcal{M}$ *is recursively saturated.*

Internal Induction

Which further principles can be conservatively added to $\mathrm{CT}^-(\mathrm{PA})$?

Definition

The axiom of internal induction, INT, is the following sentence of $\mathcal{L}_T$:

$$\forall y \forall \phi(y) \Big(\big(T\phi[0/y] \wedge \forall x \big(T\phi[\underline{x}/y] \rightarrow T\phi[\underline{x+1}/y] \big) \big) \rightarrow \forall x T\phi[\underline{x}/y] \Big)$$

It can be shown that $\mathrm{CT}^-(\mathrm{PA}) + \mathrm{INT}$ is conservative over PA. Moreover,

Internal Induction

Which further principles can be conservatively added to $\mathrm{CT}^-(\mathrm{PA})$?

Definition

The axiom of internal induction, INT, is the following sentence of $\mathcal{L}_T$:

$$\forall y \forall \phi(y) \Big(\big(T\phi[0/y] \wedge \forall x \big(T\phi[\underline{x}/y] \rightarrow T\phi[\underline{x+1}/y] \big) \big) \rightarrow \forall x T\phi[\underline{x}/y] \Big)$$

It can be shown that $\mathrm{CT}^-(\mathrm{PA}) + \mathrm{INT}$ is conservative over PA. Moreover,

Theorem (folklore, see Fischer, 2014)

$\mathrm{CT}^-(\mathrm{PA}) + \mathrm{INT}$ *has a super-exponential speed-up over* PA. *The same holds for some other reasonable truth theories with* INT.

Proving non-speed-up for truth theories

Provable conservativity

For an axiomatic theory Th, let $\mathrm{Th}\restriction_n$ denote the set of axioms of Th of logical depth at most n.

Provable conservativity

For an axiomatic theory Th, let Th $\restriction_n$ denote the set of axioms of Th of logical depth at most n.

Lemma

Let Th *be a theory extending* PA *with an* $\mathcal{NP}$ *set of axioms. Suppose that there are polynomials* $p(n), g(n)$ *such that for every* n

Provable conservativity

For an axiomatic theory Th, let $\mathrm{Th}\restriction_n$ denote the set of axioms of Th of logical depth at most n.

Lemma

Let Th *be a theory extending* PA *with an* $\mathcal{NP}$ *set of axioms. Suppose that there are polynomials* $p(n), g(n)$ *such that for every* n

Provable conservativity

For an axiomatic theory Th, let $\mathrm{Th}\restriction_n$ denote the set of axioms of Th of logical depth at most n.

Lemma

Let Th *be a theory extending* PA *with an* $\mathcal{NP}$ *set of axioms. Suppose that there are polynomials* $p(n), g(n)$ *such that for every* n

$$\| \forall\phi((\mathrm{dp}(\phi) \leq n \wedge \mathrm{Prov}_{\mathrm{Th}\restriction_n}(\phi)) \rightarrow \mathrm{Prov}_{\mathrm{PA}\restriction_{g(n)}}(\phi)) \|_{\mathrm{PA}} \leq p(n).$$

Then there is no super polynomial speed-up of Th *over* PA*.*

Proof Sketch

Suppose Th $\vdash^n \phi$ (i.e., Th proves ϕ with a proof of size n).

Proof Sketch

Suppose Th $\vdash^{n} \phi$ (i.e., Th proves ϕ with a proof of size n). Then the depth of every formula in this proof is bounded by n.

Proof Sketch

Suppose Th $\vdash^n \phi$ (i.e., Th proves ϕ with a proof of size n). Then the depth of every formula in this proof is bounded by n. Let k code this proof.

Proof Sketch

Suppose Th $\vdash^n \phi$ (i.e., Th proves ϕ with a proof of size n). Then the depth of every formula in this proof is bounded by n. Let k code this proof. Then, by the properties of the provability predicate we have

$$\mathrm{PA} \vdash^{n^{O(1)}} \mathrm{Prov}_{\mathrm{Th}\restriction_n}(\underline{k}, \underline{\phi}).$$

Proof Sketch

Suppose Th $\vdash^n \phi$ (i.e., Th proves ϕ with a proof of size n). Then the depth of every formula in this proof is bounded by n. Let k code this proof. Then, by the properties of the provability predicate we have

$$\mathsf{PA} \vdash^{n^{O(1)}} \mathsf{Prov}_{\mathsf{Th}\restriction_n}(\underline{k}, \underline{\phi}).$$

Hence also $\mathsf{PA} \vdash^{n^{O(1)}} \mathsf{Pr}_{\mathsf{Th}\restriction_n}(\underline{\phi})$.

Proof Sketch

Suppose Th $\vdash^n \phi$ (i.e., Th proves ϕ with a proof of size n). Then the depth of every formula in this proof is bounded by n. Let k code this proof. Then, by the properties of the provability predicate we have

$$\mathrm{PA} \vdash^{n^{O(1)}} \mathrm{Prov}_{\mathrm{Th}\restriction_n}(\underline{k}, \underline{\phi}).$$

Hence also $\mathrm{PA} \vdash^{n^{O(1)}} \mathrm{Pr}_{\mathrm{Th}\restriction_n}(\underline{\phi})$. By the formalized conservativity we have that $\mathrm{PA} \vdash^{n^{O(1)}} \mathrm{Pr}_{\mathrm{PA}\restriction_{g(n)}}(\underline{\phi})$.

Proof Sketch

Suppose Th $\vdash^n \phi$ (i.e., Th proves ϕ with a proof of size n). Then the depth of every formula in this proof is bounded by n. Let k code this proof. Then, by the properties of the provability predicate we have

$$\mathrm{PA} \vdash^{n^{O(1)}} \mathrm{Prov}_{\mathrm{Th}\restriction_n}(\underline{k}, \underline{\phi}).$$

Hence also PA $\vdash^{n^{O(1)}} \mathrm{Pr}_{\mathrm{Th}\restriction_n}(\underline{\phi})$. By the formalized conservativity we have that PA $\vdash^{n^{O(1)}} \mathrm{Pr}_{\mathrm{PA}\restriction_{g(n)}}(\underline{\phi})$. Since PA quickly proves reflection principles for its small fragments, we have

$$\mathrm{PA} \vdash^{n^{O(1)}} \mathrm{Tr}_n(\underline{\phi}).$$

By quickly provable Tarski conditions for Tr_n (a bit tricky!) we have that PA $\vdash^{n^{O(1)}} \phi$. All the intermediate steps were polynomial in n.

Formalized conservativity of CT^-

We will prove that there exists a $p(n)$ such that for all sufficiently big n's

$$\| \forall \phi((\mathrm{dp}(\phi) \leq n \wedge \mathrm{Prov}_{\mathrm{CT}^-(\mathrm{PA}\restriction_n)}(\phi)) \rightarrow \mathrm{Prov}_{\mathrm{PA}\restriction_n}(\phi)) \|_{\mathrm{PA}} \leq p(n).$$

To this end we formalize Enayat-Visser construction.

Formalized conservativity of CT^-

We will prove that there exists a $p(n)$ such that for all sufficiently big n's

$$\| \forall\phi((\mathrm{dp}(\phi) \leq n \wedge \mathrm{Prov}_{\mathrm{CT}^-(\mathrm{PA}\restriction_n)}(\phi)) \rightarrow \mathrm{Prov}_{\mathrm{PA}\restriction_n}(\phi)) \|_{\mathrm{PA}} \leq p(n).$$

To this end we formalize Enayat-Visser construction. The idea of how to implement it inside PA is due to Ali Enayat.

Formalized conservativity of CT^-

We will prove that there exists a $p(n)$ such that for all sufficiently big n's

$$\| \forall\phi((\mathrm{dp}(\phi) \leq n \wedge \mathrm{Prov}_{CT^-(PA\restriction_n)}(\phi)) \rightarrow \mathrm{Prov}_{PA\restriction_n}(\phi)) \|_{PA} \leq p(n).$$

To this end we formalize Enayat-Visser construction. The idea of how to implement it inside PA is due to Ali Enayat.