Piecewise linear homogeneous valued constraint satisfaction problems

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Example of (infinite domain) VCSP: Min Correlation Clustering

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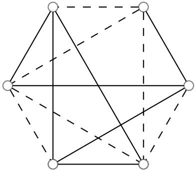

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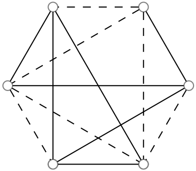

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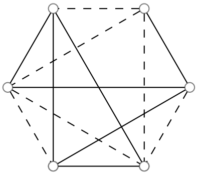

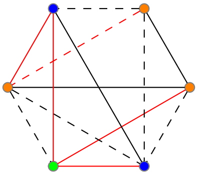

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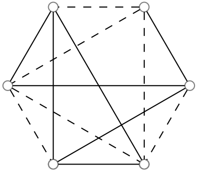

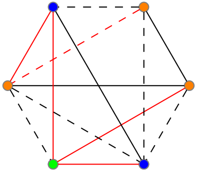

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Min Correlation Clustering is NP-hard [Bansal - Blum - Chawla, 2004].

Valued constraint languages

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Example (Valued language for Min Correlation Clustering)

$\Gamma = \{f_1, f_2\}$ where $f_1, f_2 \colon \mathbb{Q}^2 \to \{0, 1\}$ and

$$f_1(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases} \qquad f_2(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

Instances of VCSPs

Definition

An instance I of the valued constraint satisfaction problem for Γ, $VCSP(\Gamma)$, consists of

- a finite set of variables V,
- an expression ϕ of the form

$$\sum_{i=1}^{m} f_i(x_1^i, \ldots, x_{ar(f_i)}^i)$$

where $f_1, \ldots, f_m \in \Gamma$ and all the x_j^i are variables from V, and

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- a value $u \in \mathbb{Q} \cup \{+\infty\}$.

Task: decide whether there exists a map $\alpha \colon V \to D$ whose cost

$$\sum_{i=1}^{m} f_i^{\Gamma}(\alpha(x_1^i), \ldots, \alpha(x_{ar(f_i)}^i))$$

is finite, and if so, whether there is one whose cost is smaller than or equal to u.

Finite domain VCSPs

Theorem (Kolmogorov-Krokhin-Rolinek, 2015 + Bulatov-Zhuk, 2017)

The VCSP for a finite set of cost functions over a finite domain is either polynomial-time solvable or NP-hard.

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We will focus on a very natural class of infinite domain VCSPs.

A class of infinite domain VCSPs

A function is piecewise linear homogeneous (PLH) if it is first-order definable over $(\mathbb{Q}; <, 1, (c\cdot)_{c\in\mathbb{Q}})$, where $c\cdot$ denotes the map $x \mapsto cx$.

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Example (Binary PLH function)

$$f(x,y) = \begin{cases} 2y & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \wedge y \geq 3 \\ -3y & \text{if } \underbrace{x \leq 0 \wedge y < 3}_{\text{“case distinctions”}}. \end{cases}$$

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Observation: every finite domain VCSP is equivalent to a PLH VCSP.

Representing PLH cost functions

Lemma (Bodirsky-Mamino-V.,2018)

$(\mathbb{Q}; <, 1, (c\cdot)_{c\in\mathbb{Q}})$ *admits quantifier elimination.*

Consequence: every PLH cost function can be written in the form

$$f(x,y) = \begin{cases} \alpha_1 x + \beta_1 y & \chi_1(x,y) \\ \vdots & \vdots \\ \alpha_l x + \beta_l y & \chi_l(x,y) \end{cases}$$

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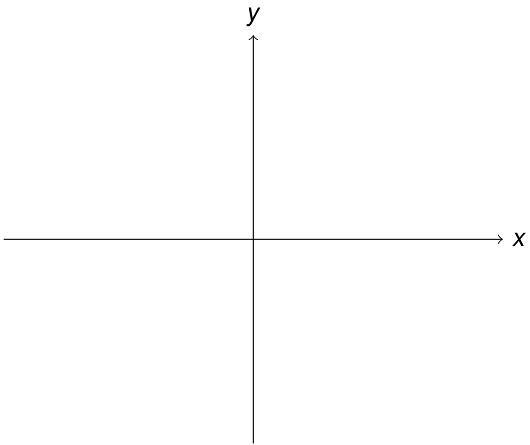
Every atomic formula is equivalent to either

- $c_1 \cdot x \overset{<}{=} c_2 \cdot y$, or
- $c_1 \cdot x \overset{<}{=} c_2 \cdot 1$, or
- $c_1 \cdot 1 \overset{<}{=} c_2 \cdot y$,

for some $c_i \in \mathbb{Q}$.

From a PLH VCSP to a finite domain VCSP: sampling



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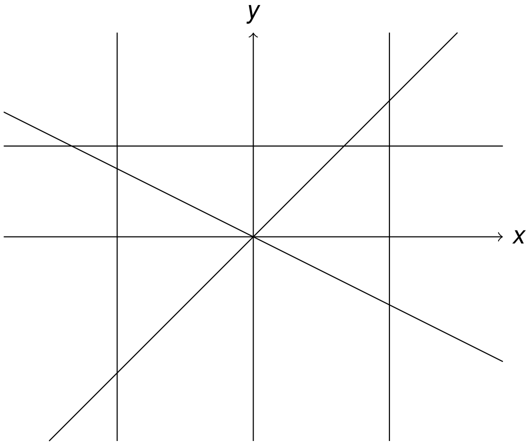

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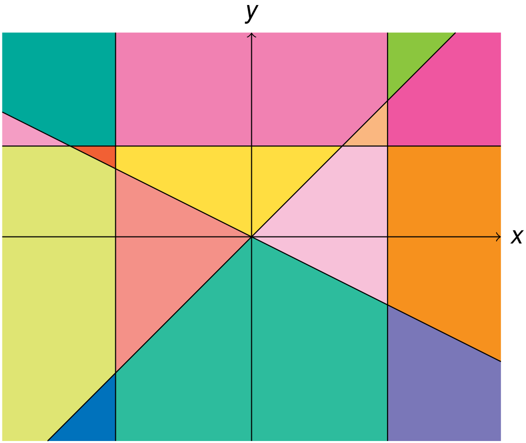

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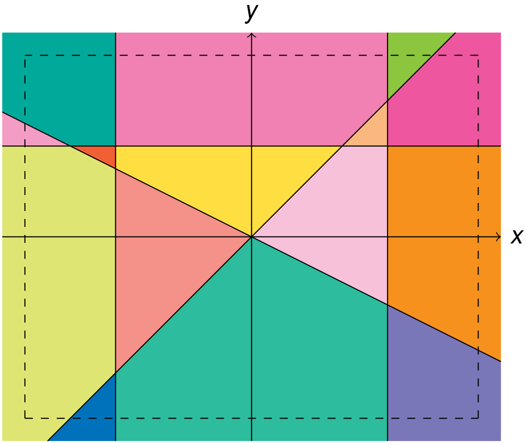

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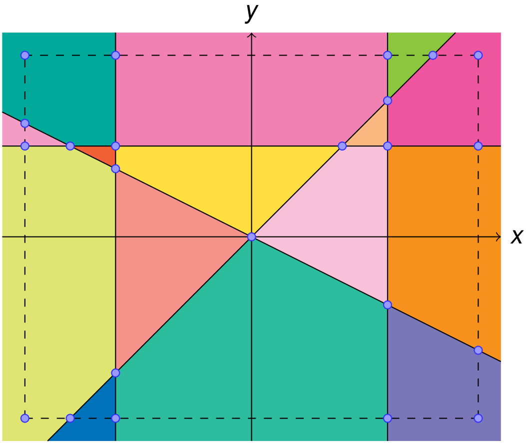

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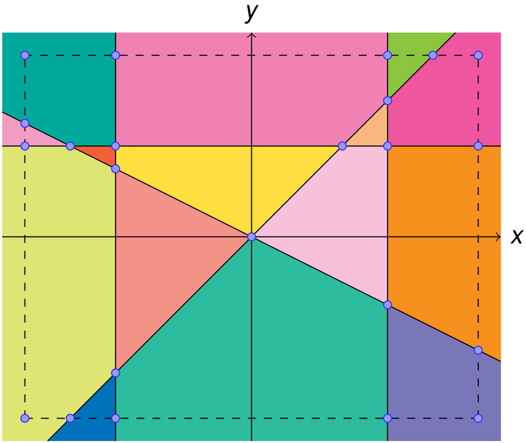

Observation: The representation of PLH cost functions suggests a "valued sampling approach".

Sampling

$\mathfrak{A}$: a structure with a finite relational signature τ.

Definition

A sampling algorithm for $\mathfrak{A}$ takes as input a positive integer *d* and computes a finite structure $\mathfrak{B}$ s.t. for every conjunction of τ-atomic formulas, χ, having at most *d* distinct free variables,

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A sampling algorithm is efficient if its running time is bounded by a polynomial in *d*.

Towards a valued sampling algorithm

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- the regions of linearity need not be closed, in general;
- the regions of linearity need not be bounded, in general;
- the sample might have super-polynomial size.

The standard trick of non-standard analysis

We interpret our cost functions in a new domain:

$$\mathbb{Q}^\star := \left\{ \sum_{i=-\infty}^{+\infty} a_i \epsilon^i \;\middle|\; a_i \in \mathbb{Q}, \text{ and } a_i \neq 0 \text{ for only finitely many negative values of } i \right\}$$

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For all $m < n$ we define the vector space

$$\mathbb{Q}^\star_{m,n} := \left\{ \sum_{i=m}^{n} a_i \epsilon^i \;\middle|\; a_i \in \mathbb{Q} \right\}$$

An efficient sampling for PLH cost functions

Theorem (Bodirsky-Mamino-V., 2018)

Let $\Phi(x_1, \ldots, x_d)$ *be a set of atomic PLH formulas. Let* $u, \alpha_1, \ldots, \alpha_d \in \mathbb{Q}$.

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such that TFAE:

1. *the formulas in* Φ *are simultaneously satisfiable in* $\mathbb{Q}$ *by a point* $(x_1, \ldots, x_d) \in \mathbb{Q}^d$ *s.t.* $\sum \alpha_i x_i \leq u$
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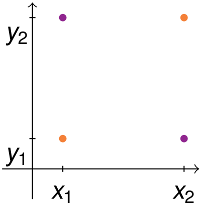
Consequence: there exists an efficient “valued sampling algorithm” for valued PLH languages.

Submodular PLH VCSPs

A function $f\colon \mathbb{Q}^n \to \mathbb{Q} \cup \{+\infty\}$ is submodular if, for all $x, y \in \mathbb{Q}^n$

$$f(x) + f(y) \geq f(\min(x, y)) + f(\max(x, y))$$

where min and max are applied componentwise.



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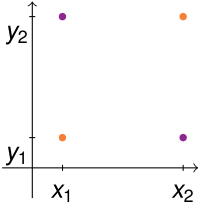

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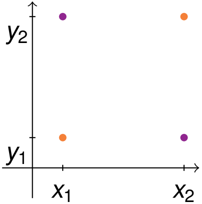

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Theorem (Bodirsky-Mamino-V., 2018)

Let Γ be a finite language of submodular PLH cost functions. Then VCSP(Γ) is solvable in polynomial-time.

Maximal tractability

Let $\mathcal{V}$ be a class of valued constraint languages over a fixed domain D and let Γ be a language of $\mathcal{V}$, Γ is maximally tractable within $\mathcal{V}$ if

- $VCSP(\Gamma') \in P$ for every finite sublanguage $\Gamma' \subset \Gamma$; and
- for every $f \in \mathcal{V}$ such that $f \notin \Gamma$, there exists a finite sublanguage $\Delta \subseteq \Gamma$ such that $VCSP(\Delta \cup \{f\})$ is NP-hard.

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The valued language consisting of all submodular PLH cost functions is maximally tractable within the class of PLH valued languages.

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In the second case we can use them (after having computed a suitable ϵ).

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