Logic and topology

some connections

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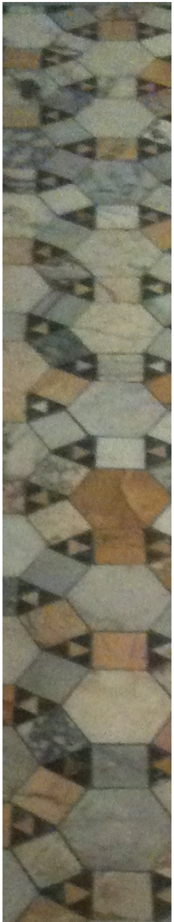
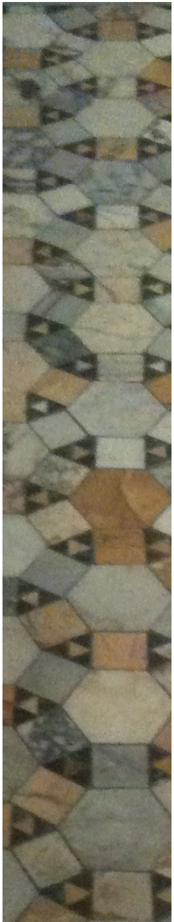


A A list of connections from the early days

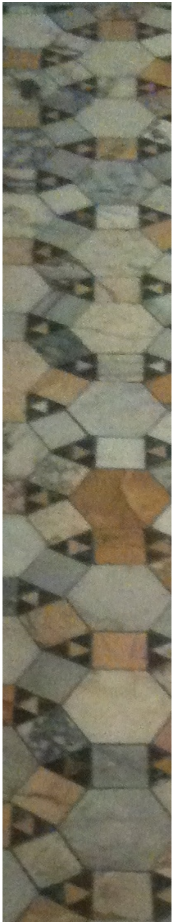


Topological models for various logic:

- Opens as interpretations of propositional intuitionistic logic. — or modal logic.
- Models of first order intuitionistic logic as (pre)sheaves of classical models.

Categorical interpretation of intuitionistic proofs

- Proofs as "Scott continuous" maps between object/structured sets/ types —– e.g. coherence spaces taht work for second order as well.
- Intuitionnistic types theory and its homotopic model.

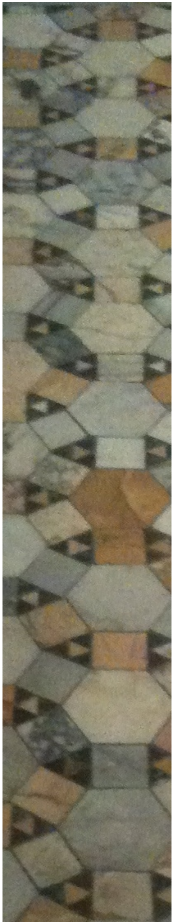


Topological methods in logic:

- Topological semantics for logic with a provability modality.
- Completion of theories endowed with a topology as a dynamical system. $T \mapsto T + Co(T)$.

Logical methods for topology:

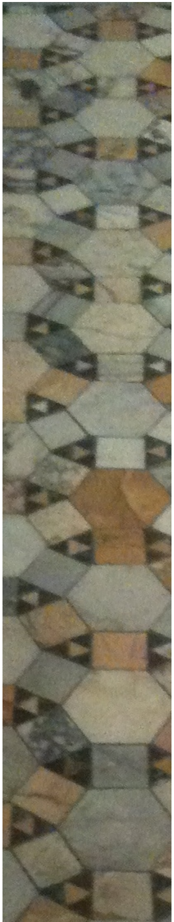
- Logics for topological and spatial description.



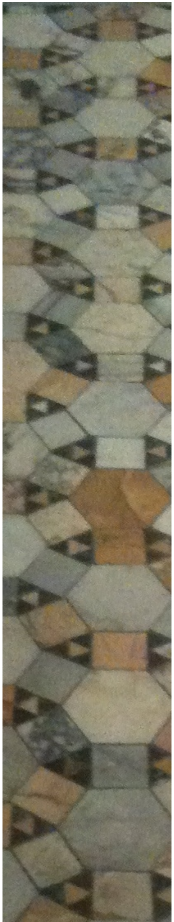
A.1. Our selection

- Opens as interpretations of propositional intuitionistic logic.
- Models of first order intuitionistic logic as (pre)sheaves of classical models.
- Topological semantics for logic with a provability modality.

Some other choices are possible... ma al bar con uno bicchiere di Ramandolo.



B Interpretations of (intuitionistic) propositions by open sets

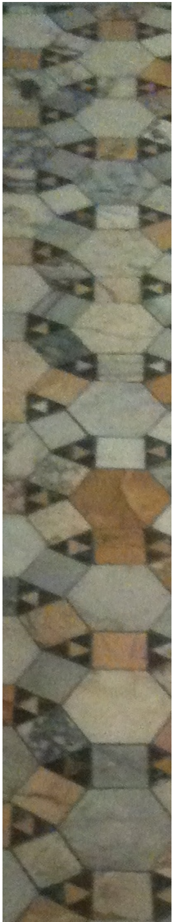


B.1. Intuitionistic propositional calculus

0-ary connective / constant: $\bot$.

Binary connectives: $\wedge, \vee, \to$
— beware that they are all independent.

$\neg P$ is defined as $P \to \bot$.



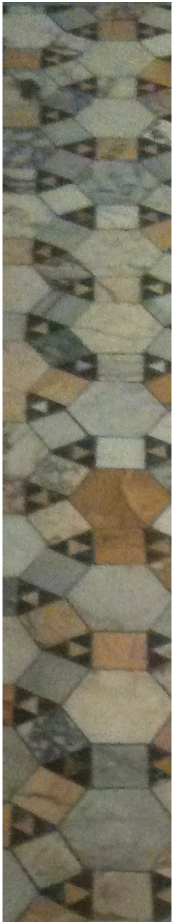
B.2. Intuitionistic natural deduction

Deduction rules as expected,
but a single conclusion!

(no contraction,
no weakening
on the conclusion side!)

Disjunction property: if $\vdash A \vee B$ then $\vdash A$ or $\vdash B$.

$P \vdash \neg\neg P$ but not the converse.



B.3. Form formulas to open sets

Given a topology $\mathcal{T}$ on X define $[\![p]\!] \in \mathcal{T}$ for any propositional variable p.

This extends to any propositional formula:

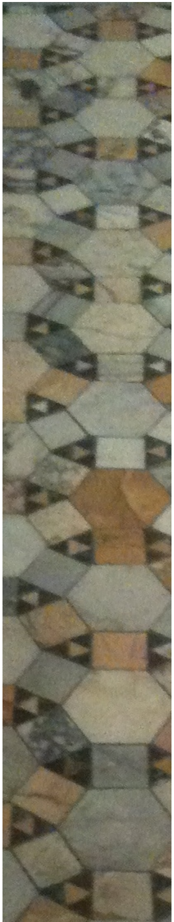
$[\![\bot]\!] = \emptyset \qquad [\![\top]\!] = X$

$[\![p \wedge q]\!] = [\![p]\!] \cap [\![q]\!]$

$[\![p \vee q]\!] = [\![p]\!] \cup [\![q]\!]$

$[\![p \rightarrow q]\!] = \bigcup_{(x \cap [\![p]\!]) \subset [\![q]\!]} x$

$F \vdash G$ true in a model whenever $[\![F]\!] \subset [\![G]\!]$.



B.4. Soundness and completeness

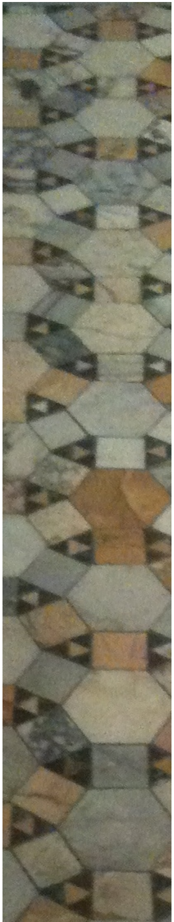
Soundness by induction on deduction rules.
Completeness from the topological space induced by trees of classical models.

Standard $\mathbb{R}$ models are enough to obtain completeness (idea: continuous map from $\mathbb{R}^n$ to any finite tree).

$[\![\neg P]\!]$ simply is the interior of $X \setminus [\![P]\!]$.

Example $\neg\neg P \vdash P$ is not provable:
define $[\![P]\!] =]0;1[\cup]1;2[$. Then $[\![\neg\neg P]\!] =]0;2[$

B.5. Comments



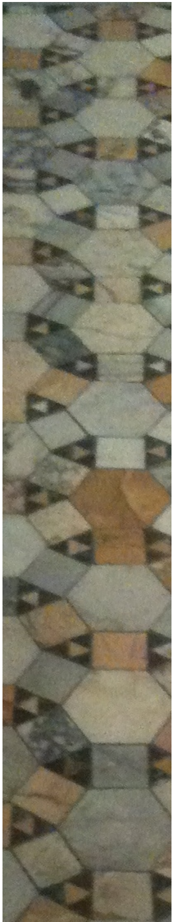
Oldest connection between logic and topology? (slightly earlier work by Tarski on S4).

Propositions up to intuitionistic equivalence: Heyting algebra lattice with a $\rightarrow$ relation satisfying some relation the order.

In a complete Heyting algebra (any subset as a sup) $\rightarrow$ can be defined as we did.

Opens of a topological space ordered with inclusion are a complete Heyting algebra.

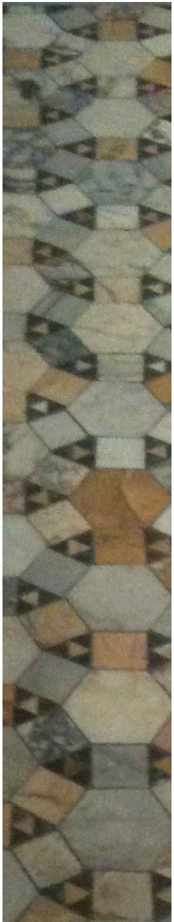
C (Pre)sheaves and intuitionistic predicate calculus



C.1. Deduction rules

Intuitionistic existential statements are stronger:
$\neg\forall x \neg P(x) \nvdash \exists x P(x)$
(rules as classical rules, limited by structural rules).

Existence property: if $\vdash \exists x.\ P(x)$ then there exists a term t such that $\vdash P(t)$

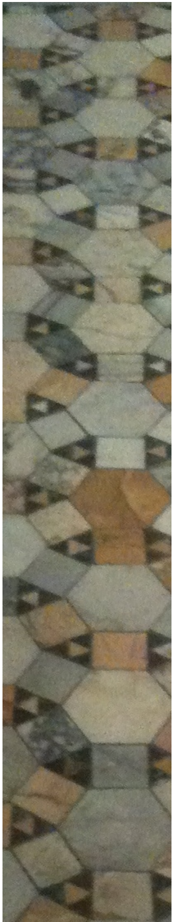


C.2. (Pre)sheaves

Idea: continuous variation of an *algebraic structure.* Assume a topology or Grothendieck topology $\mathcal{T}$.

Pre sheaf: $u \in \mathcal{T} \mapsto M_u$ is a contravariant function from the topology with inclusion morphisms to the category of structures: when $v \subset u$ there is a morphism $\rho_{u,v} : M_u \to M_v$ (functoriality: $\rho_{u_3,u_2} \circ \rho_{u_1,u_2} = \rho_{u_1,u_3}$ whenever is makes sense, i.e. $u_3 \subset u_2 \subset u_1$).

Example of pre-sheafs on $\mathbb{R}$:
$u \mapsto B(u, \mathbb{R})$ ring of bounded functions from u to $\mathbb{R}$.
$u \mapsto C(u, \mathbb{R})$ ring of continuous functions from u to $\mathbb{R}$.



C.3. Sheaves

The presheaf is said to be a sheaf if every family of compatible elements has unique glueing:

given a cover U_i of an open set U,

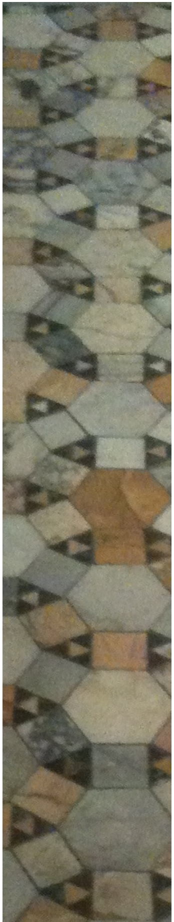
with for every i an element $c_i \in M_{u_i}$ such that

for every pair i, j $\rho_{u_i,u_j}(c_i) = \rho_{u_j,u_i}(c_j)$ there is a unique c in M_u such that $c_i = \rho_{u,u_i}(c)$.

Example of sheaf on the topological space R:
$u \mapsto C(u, R)$ the ring of continuous functions from u to R.
Presheaf, but not a sheaf $u \mapsto B(u, R)$ the ring of bounded functions from u to R.

C.4. Sheaf of $\mathcal{L}$-structures

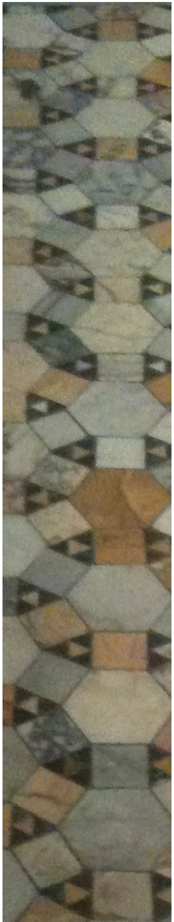
Classical model of "group" : a group.
Intuitionistic model of "group" : a sheaf of groups.
Additional property: For any $n-$ary relation symbol R, for any tuple $(a_1, \dots, a_n)$ from M_u
if for a cover u_i of u such that $(a_1^i, \dots, a_n^i) \in R_{u_i}$ for all $i \in \mathcal{I}$ then $(a_1, \dots, a_n) \in R_u$.



C.5. Presheaf semantics: Kripke-Joyal forcing — 1/3 atoms and conjunction

Formulas of $\mathcal{L}$ can be inductively interpreted on an object u of a given presheaf model M (ν: assignment into M_u):

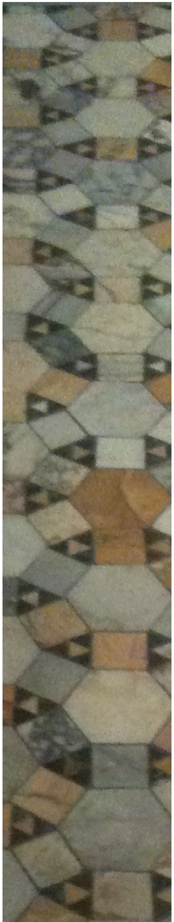
- $u \Vdash_\nu R(t_1, \ldots, t_n)$ iff $([t_1]_\nu, \ldots, [t_n]_\nu) \in R_u$.
- $u \Vdash_\nu t_1 = t_2$ iff $\quad [t_1]_\nu = [t_2]_\nu$.
- $\emptyset \Vdash_\nu \bot$
- $u \Vdash_\nu \phi \wedge \psi$ iff $u \Vdash_\nu \phi$ and $u \Vdash_\nu \psi$.



C.6. Presheaf semantics: Kripke-Joyal forcing — 2/3 disjunction and existential

Formulas of $\mathcal{L}$ can be inductively interpreted on an object u of a given presheaf model M (ν: assignment into M_u):

- $u \Vdash_\nu \phi \vee \psi$ iff there exist opens u_1, u_2 $u_1 \cup u_2 = u$ such that $u_1 \Vdash_\nu \phi$ and $u_2 \Vdash_\nu \psi$.
- $u \Vdash_\nu \exists x \phi$ iff iff there is a covering u_i of u and elements $a_i \in |M_{u_i}|$ for $i \in \mathcal{I}$ such that $u_i \Vdash_{\nu[x \mapsto a_i]} \phi$ for any index i.



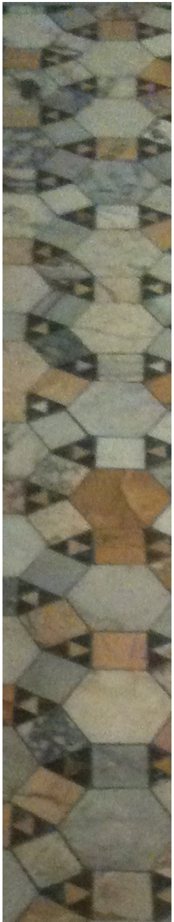
C.7. Presheaf semantics: Kripke-Joyal forcing — 3/3 implication and universal

Formulas of $\mathcal{L}$ can be inductively interpreted on an object u of a given presheaf model M (ν: assignment into M_u):

- $u \Vdash \phi \rightarrow \psi$ iff for all $v \subset u$, if $v \Vdash \phi$ then $v \Vdash \psi$.
- $u \Vdash \neg\phi$ iff for all $v \subset u$, with $v \neq \emptyset$, $v \nVdash \phi$.
- $u \Vdash_{\nu} \forall x \phi$ iff for all $v \subset u$ and $a \in M_v$, $v \Vdash_{\nu[x \mapsto a]} \phi$.

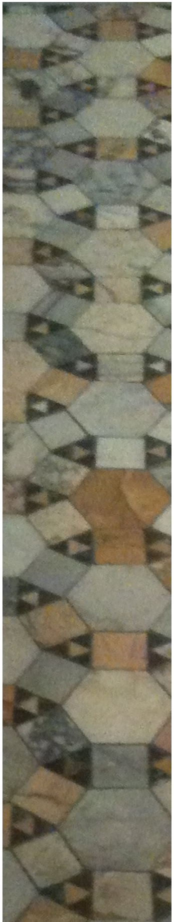
C.8. Properties of Kripke-Joyal forcing

Functoriality of $\Vdash$:
if $u_i \subset u_j$ and $u_j \Vdash F(t_1, \dots, t_n)$ then $u_i \Vdash F(t_1^i, \dots, t_n^i)$
where t_k^j is simply the restriction of t_k to u_i.
Locality of validity:
we asked for the validity of atoms to be local, but Krike-Joyal forcing propagates this property to all formulae:
If there exist a covering u_i of u and if for all i one has $u_i \Vdash F(t_1^i, \dots, t_n^i)$ then $u \Vdash F(t_1, \dots, t_n)$



C.9. Soundness and Completeness

IQC proves F iff $X \Vdash \top$
for any interpretation over a topological space X.



C.10. A remark on $C_{\mathbb{R}}$

$]a, b[\Vdash\!\!\!\!/ \ (\ell = 0 \vee \neg(\ell = 0))$
with $\ell(x) = 0$ for $x \in]a, (a+b)/2[$,
and $\ell(x) = x - (a+b)/2$ for $x \in](a+b)/2, b[$.

Indeed, otherwise there would exist u_1, u_2 with $u_1 \cup u_2 =]a, (a+b)/2[$, such that $\ell(x_1) = 0$ for all x_1 in u_1 and $\ell(x_2) \neq 0$ for all x_2 in u_2. This is impossible because $(a+b)/2$ ought to be either in u_1 or in u_2.

If $(a+b)/2$ is in u_1 then ℓ should be constantly 0 in u_1 hence around $(a+b)/2$.

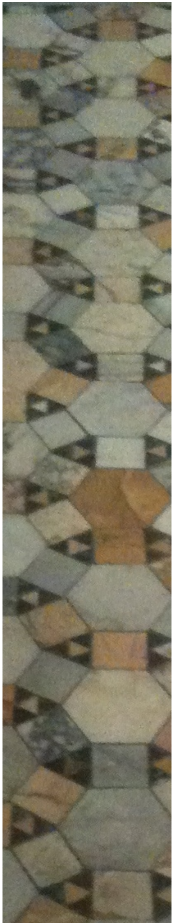
If $(a+b)/2$ is in u_2 impossible because ℓ should not be constantly 0 on any open in u_2.

C.11. Example

Let $A[f] = (f = 0 \vee \neg(f = 0))$ $C_{\mathbb{R}}$ the sheaf of rings of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ validates both:
(1) $\neg\forall f \; A[f]$
(2) $\forall f \; \neg\neg A[f]$ (provable — true in any model)

Let us see that $\neg\forall f \; A[f]$ is true in $C_{\mathbb{R}}$
$\neg\forall f \; A[f]$ is true in $C_{\mathbb{R}}$ means that
for any $u \neq \emptyset$ $u \not\Vdash \forall f \; A[f]$
i.e. there exists $v \subset u$ and f a continuous function on v such that $v \not\Vdash A[f]$.
$u \neq \emptyset$, so u contains $v =]a, b[$ — $f = \ell$ defined above shows that $\neg\forall f \; A[f]$ is true in $C_{\mathbb{R}}$.

Thus (1) $\rightarrow \neg$(2) is not true in intuitionistic logic
but $\neg$(2) and (1) are classically equivalent.



D Topological models of logic with a provability modality

D.1. Gödel-Löb logic

$\Box\varphi$: T (including PA) proves φ.
$\Diamond\varphi$: $\neg\Box\neg\varphi$
Language:

$$p \qquad \neg\varphi \qquad \varphi \wedge \psi \qquad \Box\varphi$$

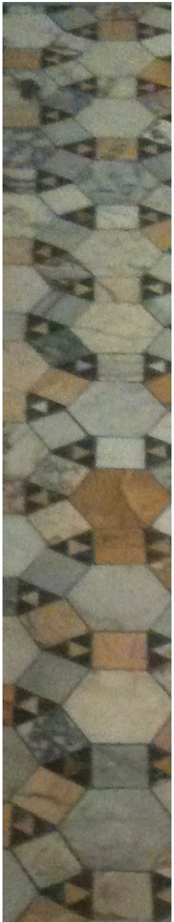
Axioms:

- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ (Löb's axiom)

Second incompleteness theorem:

$$\Box\Diamond\top \to \Box\bot$$

Sound and complete arithmetical interpretations.



D.2. Topological semantics:

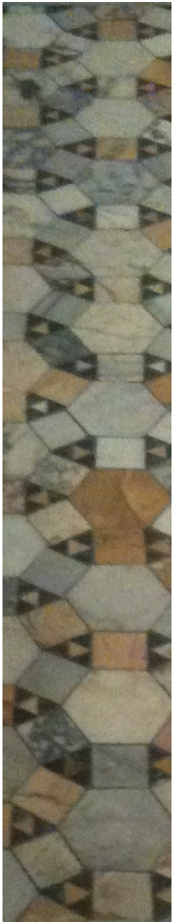
- GL-spaces: scattered topological spaces $\langle X, \mathcal{T} \rangle$

  Scattered: Every non-empty subset contains an isolated point.

- Valuations: dA is the set of limit (or accumulation) points of A.

$$[\![\Diamond\varphi]\!] = d[\![\varphi]\!]$$

GL is also sound and complete for this interpretation.

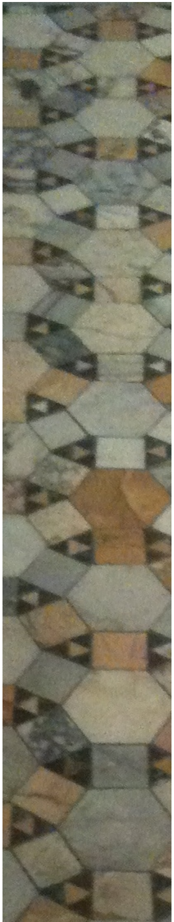


D.3. Some scattered spaces

- A finite partial order $\langle W, < \rangle$ with the downset topology
- An ordinal ξ with the initial segment topology
- An ordinal ξ with the order topology

Non-scattered:

- The real line
- The rational numbers
- The Cantor set



D.4. Ordinal numbers

Ordinals serve as canonical representatives of well-orders.

Well-order: Structure $\langle A, \preccurlyeq \rangle$ such that

- A is any set,
- $\preccurlyeq$ is a linear order on A, and
- if $B \subseteq A$ is non-empty, then it has a $\preccurlyeq$-minimal element.

The class Ord of ordinals is itself well-ordered:

$$\xi \leq \zeta \Leftrightarrow \xi \subseteq \zeta.$$

Examples:

- Every interval $[0, n)$ is an ordinal for $n \in \mathbb{N}$.
- The set of natural numbers can itself be seen as the first infinite ordinal, and is denoted ω.

D.5. Ordinal topologies

Intervals on ordinals are defined in the usual way, e.g.

$$[\alpha, \beta) = \{\xi : \alpha \leq \xi < \beta\}.$$

- Initial topologies: Topology $\mathcal{I}_0$ on an ordinal Θ generated by sets of the form $[0, \alpha)$.

- Interval topologies: Topology $\mathcal{I}_1$ on an ordinal Θ generated by sets of the form $[0, \alpha)$ and (α, β).

D.6. Iterated derived sets

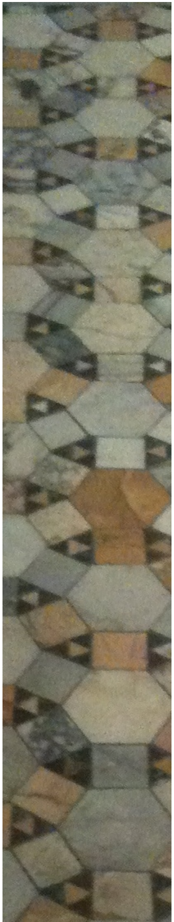
Recall that if $\langle X, \mathcal{T} \rangle$ is any topological space and $A \subseteq X$, dA denotes the set of limit points of A.

If ξ is an ordinal, define $d^\xi A$ recursively by:

1. $d^0 A = A$
2. $d^{\zeta+1} A = dd^\zeta A$
3. $d^\lambda A = \bigcap_{\zeta<\lambda} d^\zeta A$ (λ a limit).

Theorem 1. *The following are equivalent:*

- $\langle X, \mathcal{T} \rangle$ *is scattered*
- *there exists an ordinal* Λ *such that* $d^\Lambda X = \varnothing$.



D.7. Ranks on a scattered space

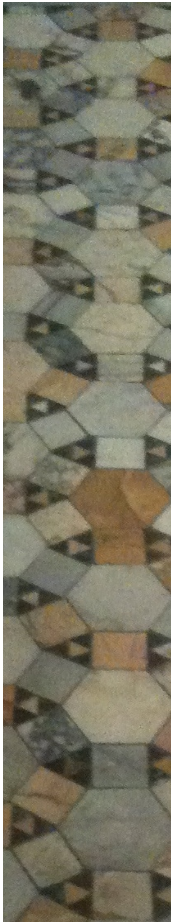
Let $\mathfrak{X} = \langle X, \mathcal{T} \rangle$ be a scattered space.

- Define $\rho(x)$ to be the least ordinal such that $x \notin d^{\rho(x)+1}X$.
- Define $\rho(\mathfrak{X})$ to be the least ordinal such that $d^{\rho(\mathfrak{X})}X = \varnothing$.

Fact: The rank on $\langle \Theta, \mathcal{I}_0 \rangle$ is the identity.

Henceforth:

- ρ_0 is the rank with respect to $\mathcal{I}_0$
- ρ_1 is the rank with respect to $\mathcal{I}_1$.



D.8. Completeness

Observation:

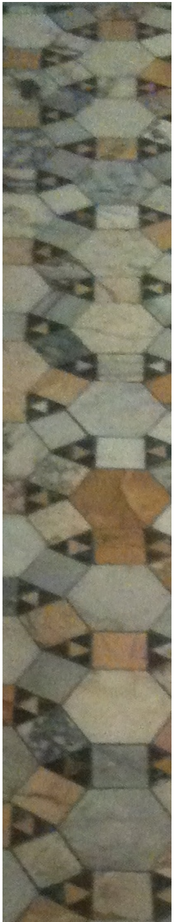
- The initial topology validates

  $$\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p).$$

- Any space of rank $n < \omega$ validates $\Box^{n+1}\bot$.

- The first ordinal with infinite ρ_1 is ω^ω.

Theorem 2 (Abashidze, Blass). *If* $\Theta \geq \omega^\omega$, *then* GL *is complete for* $\langle \Theta, \mathcal{I}_1 \rangle$*.*



E Conclusion

Quite difficult to give an overview on the connection between logic and topology in 15'.
We hope the talk will suggest some discussions during the logic colloquium.
Some of those connections are quite active, and that's the most important.

- Sheaf semantics: with topology? for modal logics (S4)? direct proofs of completeness?
- Provability logics: dynamical systems $T \to T+Con(T)$ fixpoint $=$ incoherent theories, measur-