A finitely supported frame for TSC

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Overview

- **T**uring **S**chmerl **C**alculus (TSC) is a modal system tailored to express the principles that hold between Turing progressions;

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- Strictly positive signature with no variables;

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- **T**uring **S**chmerl **C**alculus (TSC) is a modal system tailored to express the principles that hold between Turing progressions;
- Strictly positive signature with no variables;
- Inspired by **RC**, **GLP**;

Overview

- TSC is complete w.r.t. a natural arithmetical interpretation;
- TSC is complete w.r.t. a minor variation of Ignatiev Universal Frame $\mathcal{I}$;

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- TSC is complete w.r.t. a natural arithmetical interpretation;
- TSC is complete w.r.t. a minor variation of Ignatiev Universal Frame $\mathcal{I}$;
  - Special sequences of ordinals;
  - A new universal frame $\mathcal{H}$ that is based only in those sequences which have finite support.

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Turing Progressions

- By Gödel's second incompleteness theorem $T \nvdash \mathrm{Con}(T)$;

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- $T + \mathrm{Con}(T) \nvdash \mathrm{Con}(T + \mathrm{Con}(T))$;
- $T + \mathrm{Con}(T) + \mathrm{Con}(T + \mathrm{Con}(T)) \ldots$.

Turing Progressions

T1. $T^0 := T$ where T is an initial theory;

T2. $T^{\alpha+1} := T^\alpha + \mathrm{Con}(T^\alpha)$;

T3. $T^\lambda := \bigcup_{\beta \prec \lambda} T^\beta$, for λ a limit ordinal.

Graded Turing Progressions

GT1. $(T)^0_n := T$ where T is an initial theory;

GT2. $(T)^{\alpha+1}_n := (T)^\alpha_n + \mathrm{Con}_n((T)^\alpha_n)$;

GT3. $(T)^\lambda_n := \bigcup_{\beta \prec \lambda} (T)^\beta_n$, for λ a limit ordinal.

Some principles

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- **Hyperexponential functions**:
  - $e^0(\alpha) = \alpha$;
  - $e^1(\alpha) = -1 + \omega^\alpha$;
  - $e^{n+m}(\alpha) = e^n(e^m(\alpha))$.

Some principles

- **Monotonicity:** $(T)_n^\beta \subseteq (T)_n^\alpha$ for $\beta \leq \alpha$;
- **Additive Principle:** $((T)_n^\alpha)_n^\beta \equiv T_n^{\alpha+\beta}$;

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- **Additive Principle:** $((T)_n^{\alpha})_n^{\beta} \equiv T_n^{\alpha+\beta}$;
- **Reduction Property:** $(T)_{n+1}^{\alpha} \equiv_{\Pi_{n+1}} (T)_n^{e^1(\alpha)}$;
- **Reduction Property*:** $(T)_n^{e^1(\alpha)} \subseteq (T)_{n+1}^{\alpha}$;

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- **Reduction Property:** $(T)_{n+1}^\alpha \equiv_{\Pi_{n+1}} (T)_n^{e^1(\alpha)}$;
- **Reduction Property***: $(T)_n^{e^1(\alpha)} \subseteq (T)_{n+1}^\alpha$;
- **Schmerl Principle:** $((T)_{m+k}^\alpha)_m^\beta \equiv_{\Pi_{m+1}} (T)_m^{e^k(\alpha)\cdot(1+\beta)}$ $(\alpha > 0)$;
- **Schmerl Principle***: $((T)_{m+k}^\alpha)_m^\beta \equiv (T)_m^{e^k(\alpha)\cdot(1+\beta)} + (T)_{m+k}^\alpha$ $(\alpha > 0)$.

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Signature and Ordinal Modalities

- Let Λ be a fixed recursive ordinal;
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$$\mathbb{F} ::= \top \mid (\varphi \wedge \psi) \mid \langle\, n^{\alpha} \,\rangle \varphi$$

IW's

Definition

The set of *increasing worms*, denoted by IW is inductively defined as follows:

i) $\top \in \mathsf{IW}$;
ii) $\langle n^{\alpha} \rangle \top \in \mathsf{IW}$ for any $n < \omega$ and α, $0 < \alpha < \Lambda$;
iii) if $\langle n^{\alpha} \rangle A \in \mathsf{IW}$ and $m < n$, then $\langle m^{\beta} \rangle \langle n^{\alpha} \rangle A \in \mathsf{IW}$.

IW's

Definition

Let $\langle n^\alpha \rangle A \in \mathrm{IW}$, $m < n$ and $\beta < \Lambda$. By $o_m^\beta(\langle n^\alpha \rangle A)$ we denote the *m-β-ordinal* of $\langle n^\alpha \rangle A$, that is recursively defined as follows:

i) $o_m^\beta(\langle n^\alpha \rangle \top) = e^{n-m}(\alpha) \cdot (1 + \beta)$;

ii) $o_m^\beta(\langle n^\alpha \rangle A) = e^{n-m}(o_n^\alpha(A)) \cdot (1 + \beta)$.

For any $m < \omega$ and $\beta < \Lambda$, we set $o_m^\beta(\top)$ to be zero.

IW's

Definition

Let $\langle n^\alpha \rangle A \in \mathrm{IW}$, $m < n$ and $\beta < \Lambda$. By $o^\beta_m(\langle n^\alpha \rangle A)$ we denote the *m-β-ordinal* of $\langle n^\alpha \rangle A$, that is recursively defined as follows:

i) $o^\beta_m(\langle n^\alpha \rangle \top) = e^{n-m}(\alpha) \cdot (1+\beta)$;

ii) $o^\beta_m(\langle n^\alpha \rangle A) = e^{n-m}(o^\alpha_n(A)) \cdot (1+\beta)$.

For any $m < \omega$ and $\beta < \Lambda$, we set $o^\beta_m(\top)$ to be zero.

Schmerl Principle: $((T)^\alpha_{m+k})^\beta_m \equiv_{\Pi_{m+1}} (T)^{e^k(\alpha)\cdot(1+\beta)}_m \quad (\alpha > 0)$

TSC Axioms

1. $\varphi \vdash \varphi, \quad \varphi \vdash \top;$
2. $\varphi \wedge \psi \vdash \varphi, \quad (\psi);$
3. $\langle n^{\alpha} \rangle \varphi \vdash \langle n^{\beta} \rangle \varphi, \quad$ for $\beta \leq \alpha;$
4. $\langle n^{\alpha+\beta} \rangle \varphi \equiv \langle n^{\beta} \rangle \langle n^{\alpha} \rangle \varphi;$
5. $\langle m + n^{\alpha} \rangle \varphi \vdash \langle m^{e^{n}(\alpha)} \rangle \varphi;$
6. $\langle n^{\alpha} \rangle A \equiv \langle n^{o_n^{\alpha}(A)} \rangle \top \ \wedge \ A \quad$ for $\langle n^{\alpha} \rangle A \in \mathrm{IW}.$

TSC Rules

1. $\varphi \vdash \psi$ and $\phi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;
2. $\varphi \vdash \psi$ and $\psi \vdash \chi$ then $\varphi \vdash \chi$;
3. If $\varphi \vdash \psi$, then $\langle n^{\alpha} \rangle \varphi \vdash \langle n^{\alpha} \rangle \psi$;
4. $\varphi \vdash \psi$ then
   $\langle n^{\alpha} \rangle \varphi \ \wedge \ \langle m^{\beta+1} \rangle \psi \vdash \langle n^{\alpha} \rangle (\varphi \ \wedge \ \langle m^{\beta+1} \rangle \psi) \quad$ for $m < n$

Arithmetical Interpretation

- We can define a translation τ between modal formulas and arithmetical formulas numerating the axioms of Turing progressions.

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- By Th_φ we denote the arithmetical theory numerated by $\tau(\varphi)$.

TSC

Theorem (Normal Form)

For every formula φ, there is a unique $A \in \mathrm{IW}$ such that $\varphi \equiv A$.

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For every formula φ, there is a unique $A \in \mathsf{IW}$ such that $\varphi \equiv A$.

Theorem (Completeness)

For any $\varphi, \psi \in \mathcal{L}_{\mathbb{F}}$,

$$EA^{+} \vdash \ \mathsf{Th}_{\psi} \subseteq \mathsf{Th}_{\varphi} \ \ \textit{iff} \ \ \varphi \vdash \psi.$$

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Ignatiev sequences

Definition

We define *ordinal logarithm* as $lg(0) := 0$ and $lg(\alpha + \omega^{\beta}) := \beta$.

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Definition

By lg^{ω} we denote the set of *ℓ-sequences* or *Ignatiev sequences*. That is, the set of sequences $x := \langle x_0, x_1, x_2, \ldots \rangle$ where for $i < \omega$, $x_{i+1} \leq lg(x_i)$.

$\mathcal{J}$

We consider a minor variation on Ignatiev's frame.

Definition

$\mathcal{J} := \langle I, \{R_n\}_{n<\omega} \rangle$, is defined as follows:

$$I := \{x \in \mathrm{Ig}^{\omega} : x_i < \Lambda \text{ for } i < \omega\};$$

$$xR_ny :\Leftrightarrow (\forall m \leq n \; x_m > y_m \;\wedge\; \forall i > n \; x_i \geq y_i).$$

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$$xR_n y :\Leftrightarrow (\forall m \leq n\ x_m > y_m \ \wedge\ \forall i > n\ x_i \geq y_i).$$

Definition

Given $x, y \in I$ and R_n on I, we recursively define $xR_n^{\alpha}y$ as follows:

1. $xR_n^0 y \quad :\Leftrightarrow \quad x = y;$
2. $xR_n^{1+\alpha} y \quad :\Leftrightarrow\ \forall\, \beta < 1+\alpha\ \exists z\ \left(xR_n z\ \wedge\ zR_n^{\beta} y\right).$

Completeness

Theorem

TSC *is sound and complete w.r.t.* $\mathcal{J}$ *i.e*

$$\varphi \vdash \psi \quad \textit{iff} \quad \forall x \in I \, (\mathcal{J}, x \Vdash \varphi \ \Rightarrow \ \mathcal{J}, x \Vdash \psi).$$

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Finite support

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Finite support

- For $\Lambda > \varepsilon_0$, we have Ignatiev sequences that never reach zero;
- Define a new universal frame where we only consider sequences with finite support.

$\mathcal{H}$

Definition

$\mathcal{H} := \langle H, \{S_n\}_{n<\omega} \rangle$, is defined as follows:

$$H := \{x \in I : x_i = 0 \text{ for some } i < \omega\};$$

$$xS_ny :\Leftrightarrow (\forall m \leq n\ x_m > y_m\ \wedge\ \forall i > n\ x_i \geq y_i).$$

Definition

Given $x, y \in H$ and S_n on H, we recursively define $xS_n^{\alpha}y$ as follows:

1. $xS_n^0y \quad :\Leftrightarrow \quad x = y;$
2. $xS_n^{1+\alpha}y \ :\Leftrightarrow\ \forall\,\beta{<}1{+}\alpha\ \exists z\ \left(xS_nz\ \wedge\ zS_n^{\beta}y\right).$

$\mathcal{H}$

Definition

Let $x \in H$ and $\varphi \in \mathbb{F}$. By $x \Vdash \varphi$ we denote the validity of φ in x that is recursively defined as follows:

- $x \Vdash \top$ for all $x \in H$;
- $x \Vdash \varphi \wedge \psi$ iff $x \Vdash \varphi$ and $x \Vdash \psi$;
- $x \Vdash \langle n^\alpha \rangle \varphi$ iff there is $y \in H$, $x S_n^\alpha y$ and $y \Vdash \varphi$.

Completeness

Theorem

For any $x \in H$ *and* $\varphi \in \mathbb{F}$,

$$\mathcal{J}, x \Vdash \varphi \iff \mathcal{H}, x \Vdash \varphi.$$

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Theorem

For any $x \in H$ *and* $\varphi \in \mathbb{F}$,

$$\mathcal{J}, x \Vdash \varphi \iff \mathcal{H}, x \Vdash \varphi.$$

Theorem

For any φ, $\psi \in \mathbb{F}$, *we have that:*

$$\varphi \vdash \psi \iff \forall x \in H \left(\mathcal{H}, x \Vdash \varphi \implies \mathcal{H}, x \Vdash \psi \right).$$

Definability

- For $x \in H$ we define $x^{\downarrow} := \{y : y_j < x_j \text{ for some } j < \omega\}$.

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Theorem

For any $x \in H$ there is a unique $A \in \mathrm{IW}$ such that:

$$\mathcal{H}, x \Vdash A \ \ \& \ \ \forall\, y \in x^{\downarrow},\ \mathcal{H}, y \nVdash A.$$

Definability

- We can associate to any $x \in H$ an increasing worm A_x.

Proposition

For any $x, y \in H$:

$$xS_n^{\alpha}y \iff EA^+ \vdash \mathrm{Th}_{\langle n^{\alpha} \rangle A_y} \subseteq \mathrm{Th}_{A_x}.$$

Thanks for coming!