### A finitely supported frame for TSC

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Logic Colloquium - 2018

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# Contents

#### 1 Overview

- 2 Turing Progressions
  - Turing Progressions
  - Graded Turing Progressions
  - Some Principles

### 3 TSC

 Signature and Ordinal Modalities

- TSC
- 4 Universal Frame  $\mathcal J$ 
  - Ignatiev sequences
  - J
  - Completeness
- Sequences with Finite Support
  - *H*
  - Completeness

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Definability

#### Overview

Turing Progressions TSC Universal Frame  ${\mathcal J}$  Sequences with Finite Support

# Contents

#### 1 Overview

- 2 Turing Progressions
  - Turing Progressions
  - Graded Turing
    - Progressions
  - Some Principles

### 3 TSC

 Signature and Ordinal Modalities

- TSC
- 4 Universal Frame  $\mathcal J$ 
  - Ignatiev sequences
  - J
  - Completeness
- 5 Sequences with Finite Support
  - *H*
  - Completeness

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Definability



 Turing Schmerl Calculus (TSC) is a modal system tailored to express the principles that hold between Turing progressions;



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Strictly positive signature with no variables;



 Turing Schmerl Calculus (TSC) is a modal system tailored to express the principles that hold between Turing progressions;

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- Strictly positive signature with no variables;
- Inspired by RC, GLP;



- TSC is complete w.r.t. a natural arithmetical interpretation;
- TSC is complete w.r.t. a minor variation of Ignatiev Universal Frame *I*;

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- TSC is complete w.r.t. a natural arithmetical interpretation;
- TSC is complete w.r.t. a minor variation of Ignatiev Universal Frame *I*;
  - Special sequences of ordinals;
  - A new universal frame  $\mathcal{H}$  that is based only in those sequences which have finite support.

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Turing Progressions Graded Turing Progressions Some Principles

# Contents

#### 1 Overview

### 2 Turing Progressions

- Turing Progressions
- Graded Turing
  - Progressions
- Some Principles

### 3 TSC

 Signature and Ordinal Modalities

- TSC
- 4 Universal Frame  $\mathcal J$ 
  - Ignatiev sequences
  - J
  - Completeness
- 5 Sequences with Finite Support
  - *H*
  - Completeness

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Definability

Turing Progressions Graded Turing Progressions Some Principles

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# **Turing Progressions**

#### By Gödel's second incompleteness theorem $T \not\vdash Con(T)$ ;

Turing Progressions Graded Turing Progressions Some Principles

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# **Turing Progressions**

By Gödel's second incompleteness theorem  $T \not\vdash Con(T)$ ;

### T + Con(T)

Turing Progressions Graded Turing Progressions Some Principles

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# **Turing Progressions**

- By Gödel's second incompleteness theorem  $T \not\vdash Con(T)$ ;
- $T + \operatorname{Con}(T) \not\vdash \operatorname{Con}(T + \operatorname{Con}(T));$

Turing Progressions Graded Turing Progressions Some Principles

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# **Turing Progressions**

- By Gödel's second incompleteness theorem  $T \not\vdash Con(T)$ ;
- $T + \operatorname{Con}(T) \not\vdash \operatorname{Con}(T + \operatorname{Con}(T));$
- $T + \operatorname{Con}(T) + \operatorname{Con}(T + \operatorname{Con}(T)) \dots$

Turing Progressions Graded Turing Progressions Some Principles

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# **Turing Progressions**

T1. 
$$T^0 := T$$
 where T is an initial theory;

T2. 
$$T^{\alpha+1} := T^{\alpha} + \operatorname{Con}(T^{\alpha});$$

T3. 
$$T^{\lambda} := \bigcup_{\beta \prec \lambda} T^{\beta}$$
, for  $\lambda$  a limit ordinal.

Turing Progressions Graded Turing Progressions Some Principles

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### Graded Turing Progressions

GT1. 
$$(T)_n^0 := T$$
 where T is an initial theory;

GT2. 
$$(T)_n^{\alpha+1} := (T)_n^{\alpha} + \text{Con}_n((T)_n^{\alpha});$$

GT3.  $(T)_n^{\lambda} := \bigcup_{\beta \prec \lambda} (T)_n^{\beta}$ , for  $\lambda$  a limit ordinal.

Turing Progressions Graded Turing Progressions Some Principles

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# Some principles

Some principles make use of ordinal arithmetic;

Turing Progressions Graded Turing Progressions Some Principles

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- Some principles make use of ordinal arithmetic;
- Hyperexponential functions:

• 
$$e^{0}(\alpha) = \alpha;$$
  
•  $e^{1}(\alpha) = -1 + \omega^{\alpha};$   
•  $e^{n+m}(\alpha) = e^{n}(e^{m}(\alpha)).$ 

Turing Progressions Graded Turing Progressions Some Principles

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- Monotonicity:  $(T)_n^{\beta} \subseteq (T)_n^{\alpha}$  for  $\beta \leq \alpha$ ;
- Additive Principle:  $((T)_n^{\alpha})_n^{\beta} \equiv T_n^{\alpha+\beta}$ ;

Turing Progressions Graded Turing Progressions Some Principles

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- Monotonicity:  $(T)_n^{\beta} \subseteq (T)_n^{\alpha}$  for  $\beta \leq \alpha$ ;
- Additive Principle:  $((T)_n^{\alpha})_n^{\beta} \equiv T_n^{\alpha+\beta}$ ;
- **Reduction Property:**  $(T)_{n+1}^{\alpha} \equiv_{\prod_{n+1}} (T)_n^{e^1(\alpha)}$ ;
- **Reduction Property**<sup>\*</sup>:  $(T)_n^{e^1(\alpha)} \subseteq (T)_{n+1}^{\alpha}$ ;

Turing Progressions Graded Turing Progressions Some Principles

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- Monotonicity:  $(T)_n^{\beta} \subseteq (T)_n^{\alpha}$  for  $\beta \leq \alpha$ ;
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- **Reduction Property**<sup>\*</sup>:  $(T)_n^{e^1(\alpha)} \subseteq (T)_{n+1}^{\alpha}$ ;
- Schmerl Principle:  $((T)_{m+k}^{\alpha})_m^{\beta} \equiv_{\prod_{m+1}} (T)_m^{e^k(\alpha) \cdot (1+\beta)}$  $(\alpha > 0);$
- Schmerl Principle\*:  $((T)_{m+k}^{\alpha})_{m}^{\beta} \equiv (T)_{m}^{e^{k}(\alpha) \cdot (1+\beta)} + (T)_{m+k}^{\alpha}$  $(\alpha > 0).$

Signature and Ordinal Modalities TSC

# Contents

#### 1 Overview

- 2 Turing Progressions
  - Turing Progressions
  - Graded Turing
    - Progressions
  - Some Principles

### 3 TSC

 Signature and Ordinal Modalities

- TSC
- 4 Universal Frame  $\mathcal J$ 
  - Ignatiev sequences
  - J
  - Completeness
- 5 Sequences with Finite Support
  - *H*
  - Completeness

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Definability

Signature and Ordinal Modalities TSC

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# Signature and Ordinal Modalities

- Let Λ be a fixed recursive ordinal;
- Ordinal modalities: Modal connectives of the form ⟨ n<sup>α</sup> ⟩ where n < ω and α < Λ.</p>

Signature and Ordinal Modalities TSC

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# Signature and Ordinal Modalities

- Let Λ be a fixed recursive ordinal;
- Ordinal modalities: Modal connectives of the form ⟨ n<sup>α</sup> ⟩ where n < ω and α < Λ.</p>

$$\mathbb{F} ::= \top \mid (\varphi \land \psi) \mid \langle n^{\alpha} \rangle \varphi$$

IW's

#### Definition

The set of *increasing worms*, denoted by IW is inductively defined as follows:

Signature and Ordinal Modalities

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i) \top \in IW;

ii) \langle n^{\alpha} \rangle \top \in IW for any n < \omega and \alpha, 0 < \alpha < \Lambda;

iii) if \langle n^{\alpha} \rangle A \in IW and m < n, then \langle m^{\beta} \rangle \langle n^{\alpha} \rangle A \in IW.
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# IW's

#### Definition

Let  $\langle n^{\alpha} \rangle A \in IW$ , m < n and  $\beta < \Lambda$ . By  $o_m^{\beta}(\langle n^{\alpha} \rangle A)$  we denote the *m*- $\beta$ -ordinal of  $\langle n^{\alpha} \rangle A$ , that is recursively defined as follows: i)  $o_m^{\beta}(\langle n^{\alpha} \rangle \top) = e^{n-m}(\alpha) \cdot (1+\beta);$ ii)  $o_m^{\beta}(\langle n^{\alpha} \rangle A) = e^{n-m}(o_n^{\alpha}(A)) \cdot (1+\beta).$ For any  $m < \omega$  and  $\beta < \Lambda$ , we set  $o_m^{\beta}(\top)$  to be zero.

Signature and Ordinal Modalities TSC

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# IW's

#### Definition

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Schmerl Principle: 
$$((T)_{m+k}^{\alpha})_{m}^{\beta} \equiv_{\Pi_{m+1}} (T)_{m}^{e^{k}(\alpha) \cdot (1+\beta)} (\alpha > 0)$$

Signature and Ordinal Modalities TSC

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# **TSC** Axioms

1 
$$\varphi \vdash \varphi, \quad \varphi \vdash \top;$$
  
2  $\varphi \land \psi \vdash \varphi, \quad (\psi);$   
3  $\langle n^{\alpha} \rangle \varphi \vdash \langle n^{\beta} \rangle \varphi, \quad \text{for } \beta \leq \alpha;$   
4  $\langle n^{\alpha+\beta} \rangle \varphi \equiv \langle n^{\beta} \rangle \langle n^{\alpha} \rangle \varphi;$   
5  $\langle m+n^{\alpha} \rangle \varphi \vdash \langle m^{e^{n}(\alpha)} \rangle \varphi;$   
6  $\langle n^{\alpha} \rangle A \equiv \langle n^{o^{\alpha}_{n}(A)} \rangle \top \land A \quad \text{for } \langle n^{\alpha} \rangle A \in \text{IW}.$ 

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# **TSC** Rules

1 
$$\varphi \vdash \psi$$
 and  $\phi \vdash \chi$ , then  $\varphi \vdash \psi \land \chi$ ;

2 
$$\varphi \vdash \psi$$
 and  $\psi \vdash \chi$  then  $\varphi \vdash \chi$ ;

3 If 
$$\varphi \vdash \psi$$
, then  $\langle n^{\alpha} \rangle \varphi \vdash \langle n^{\alpha} \rangle \psi$ ;

$$\begin{array}{ccc} 4 & \varphi \vdash \psi \text{ then} \\ & \langle n^{\alpha} \rangle \varphi & \wedge & \langle m^{\beta+1} \rangle \psi \vdash & \langle n^{\alpha} \rangle (\varphi \wedge & \langle m^{\beta+1} \rangle \psi) & \text{ for } m < n \end{array}$$

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### Arithmetical Interpretation

We can define a translation \(\tau\) between modal formulas and arithmetical formulas numerating the axioms of Turing progressions.

Signature and Ordinal Modalities TSC

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### Arithmetical Interpretation

- We can define a translation τ between modal formulas and arithmetical formulas numerating the axioms of Turing progressions.
- By Th<sub> $\varphi$ </sub> we denote the arithmetical theory numerated by  $\tau(\varphi)$ .

Signature and Ordinal Modalities TSC

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Theorem (Normal Form)

For every formula  $\varphi$ , there is a unique  $A \in IW$  such that  $\varphi \equiv A$ .

Signature and Ordinal Modalities TSC

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#### Theorem (Normal Form)

For every formula  $\varphi$ , there is a unique  $A \in IW$  such that  $\varphi \equiv A$ .

Theorem (Completeness)

For any  $\varphi, \psi \in \mathcal{L}_{\mathbb{F}}$ ,

$$EA^+ \vdash Th_{\psi} \subseteq Th_{\varphi} \quad iff \quad \varphi \vdash \psi.$$

lgnatiev sequences  $\mathcal J$ Completeness

# Contents

#### 1 Overview

- 2 Turing Progressions
  - Turing Progressions
  - Graded Turing
    - Progressions
  - Some Principles

### 3 TSC

 Signature and Ordinal Modalities

#### TSC

### 4 Universal Frame $\mathcal J$

- Ignatiev sequences
  .7
- Completeness
- **5** Sequences with Finite Support
  - *H*
  - Completeness

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Definability

 $\begin{array}{l} \text{Ignatiev sequences} \\ \mathcal{J} \\ \text{Completeness} \end{array}$ 

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### Ignatiev sequences

#### Definition

We define ordinal logarithm as lg(0) := 0 and  $lg(\alpha + \omega^{\beta}) := \beta$ .

 $\begin{array}{l} \text{Ignatiev sequences} \\ \mathcal{J} \\ \text{Completeness} \end{array}$ 

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### Ignatiev sequences

#### Definition

We define ordinal logarithm as lg(0) := 0 and  $lg(\alpha + \omega^{\beta}) := \beta$ .

#### Definition

By  $\lg^{\omega}$  we denote the set of  $\ell$ -sequences or Ignatiev sequences. That is, the set of sequences  $x := \langle x_0, x_1, x_2, ... \rangle$  where for  $i < \omega$ ,  $x_{i+1} \leq \lg(x_i)$ .

Ignatiev sequences  $\mathcal{J}$ Completeness

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 $\mathcal{J}$ 

We consider a minor variation on Ignatiev's frame.

#### Definition

 $\mathcal{J} := \langle I, \{R_n\}_{n < \omega} \rangle$ , is defined as follows:

$$I := \{ x \in \lg^{\omega} : x_i < \Lambda \text{ for } i < \omega \};$$

$$xR_ny :\Leftrightarrow (\forall m \leq n \ x_m > y_m \land \forall i > n \ x_i \geq y_i).$$

Ignatiev sequences  $\mathcal{J}$ Completeness

 $\mathcal{J}$ 

We consider a minor variation on Ignatiev's frame.

#### Definition

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, is defined as follows:

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#### Definition

Given  $x, y \in I$  and  $R_n$  on I, we recursively define  $xR_n^{\alpha}y$  as follows: **1**  $xR_n^0y$  : $\Leftrightarrow$  x = y; **2**  $xR_n^{1+\alpha}y$  : $\Leftrightarrow \forall \beta < 1+\alpha \exists z (xR_nz \land zR_n^{\beta}y)$ .

 $\begin{array}{l} \text{Ignatiev sequences} \\ \mathcal{J} \\ \text{Completeness} \end{array}$ 

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### Completeness

#### Theorem

**TSC** is sound and complete w.r.t.  $\mathcal{J}$  i.e

$$\varphi \vdash \psi \quad iff \quad \forall x \in I \left( \mathcal{J}, x \Vdash \varphi \ \Rightarrow \ \mathcal{J}, x \Vdash \psi \right).$$

 ${\cal H}$ Completeness Definability

# Contents

#### 1 Overview

- 2 Turing Progressions
  - Turing Progressions
  - Graded Turing
    - Progressions
  - Some Principles

### 3 TSC

 Signature and Ordinal Modalities

- TSC
- 4 Universal Frame  ${\cal J}$ 
  - Ignatiev sequences *T*
  - Ĵ
  - Completeness
- Sequences with Finite Support
  - *H*
  - Completeness

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Definability

H Completeness Definability

### Finite support

#### • For $\Lambda > \varepsilon_0$ , we have Ignatiev sequences that never reach zero;

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 ${\cal H}$ Completeness Definability



For  $\Lambda > \varepsilon_0$ , we have Ignatiev sequences that never reach zero;

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 Define a new universal frame where we only consider sequences with finite support.

H Completeness Definability

#### Definition

 $\mathcal{H} := \langle H, \{S_n\}_{n < \omega} \rangle$ , is defined as follows:

$$H := \{ x \in I : x_i = 0 \text{ for some } i < \omega \};$$

$$xS_ny :\Leftrightarrow (\forall m \le n \ x_m > y_m \land \forall i > n \ x_i \ge y_i).$$

#### Definition

Given  $x, y \in H$  and  $S_n$  on H, we recursively define  $xS_n^{\alpha}y$  as follows:

$$xS_n^0y :\Leftrightarrow x = y; xS_n^{1+\alpha}y :\Leftrightarrow \forall \beta < 1+\alpha \exists z (xS_nz \land zS_n^\beta y).$$

H Completeness Definability

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#### Definition

Let  $x \in H$  and  $\varphi \in \mathbb{F}$ . By  $x \Vdash \varphi$  we denote the validity of  $\varphi$  in x that is recursively defined as follows:

$$x \Vdash \top$$
 for all  $x \in H$ ;

• 
$$x \Vdash \varphi \land \psi$$
 iff  $x \Vdash \varphi$  and  $x \Vdash \psi$ ;

•  $x \Vdash \langle n^{\alpha} \rangle \varphi$  iff there is  $y \in H$ ,  $xS_n^{\alpha}y$  and  $y \Vdash \varphi$ .

H Completeness Definability

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### Completeness

#### Theorem

For any  $x \in H$  and  $\varphi \in \mathbb{F}$ ,

#### $\mathcal{J}, x \Vdash \varphi \iff \mathcal{H}, x \Vdash \varphi.$

H Completeness Definability

### Completeness

#### Theorem

For any 
$$x \in H$$
 and  $\varphi \in \mathbb{F}$ ,

$$\mathcal{J}, x \Vdash \varphi \iff \mathcal{H}, x \Vdash \varphi.$$

#### Theorem

For any  $\varphi, \psi \in \mathbb{F}$ , we have that:

$$arphidash \psi \iff orall x \in H\left(\mathcal{H}, x \Vdash arphi \Longrightarrow \mathcal{H}, x \Vdash \psi
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H Completeness Definability

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# Definability

### • For $x \in H$ we define $x^{\downarrow} := \{y : y_j < x_j \text{ for some } j < \omega\}.$

H Completeness Definability

# Definability

For 
$$x \in H$$
 we define  $x^{\downarrow} := \{y : y_j < x_j \text{ for some } j < \omega\}.$ 

#### Theorem

For any  $x \in H$  there is a unique  $A \in IW$  such that:

$$\mathcal{H}, x \Vdash A \& \forall y \in x^{\downarrow}, \mathcal{H}, y \nvDash A.$$

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H Completeness Definability



• We can associate to any  $x \in H$  an increasing worm  $A_x$ .

#### Proposition

For any  $x, y \in H$ :

$$xS_n^{\alpha}y \iff EA^+ \vdash \operatorname{Th}_{\langle n^{\alpha} \rangle A_v} \subseteq \operatorname{Th}_{A_x}.$$

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H Completeness Definability

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#### Thanks for coming!