The strange case of Goodman's conservation result

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Goodman's theorem

Goodman's theorem (Nicolas D. Goodman, 1976) says that intuitionistic arithmetic in all finite types HA^ω plus the axiom of choice AC is conservative over Heyting arithmetic HA.

(HA^ω is a theory of finite-type functionals. Essentially, Gödel's T with quantifiers. HA^ω has the same strength of Heyting arithmetic HA.)

In contrast, classical arithmetic in all finite types PA^ω plus choice (already $AC^{0,0}$) is as strong as full second-order arithmetic.

Remark: Goodman's theorem does not apply to subsystems of HA^ω with restricted induction (Ulrich Kohlenbach, 1999).

Remark: adding quantifier-free choice QF-AC to PA^ω can be done conservatively over PA (Kohlenbach, 1992).

Goodman's theorem

Goodman's Theorem

$HA^\omega + AC + RDC$ is conservative over HA.

There are several proofs of this result.

Original proof: Goodman, *The theory of the Gödel functionals*, **J. Symbolic Logic**, 1976. Based on his arithmetic theory of constructions. Regarded as complicated.

Second proof (arguably the best): Goodman, *Relativized realizability in intuitionistic arithmetic of all finite types*, **J. Symbolic Logic**, 1978. Based on "a new notion of realizability" which combines

- Kleene recursive realizability
- the model HRO of the hereditarily recursive operations
- Kripke semantics

Goodman's theorem

Other proofs: Michael Beeson (1979), Renardel de Lavalette (1990), Thierry Coquand (2013) and, more recently, Benno van den Berg and Lotte van Slotte (2017).

Goodman's theorem (extensional)

Goodman's Theorem (extensional)

E-HA$^\omega$ + AC is conservative over HA.

Problem 38 in Friedman's *One Hundred and Two Problems in Mathematical Logic*, **J. Symbolic Logic**, 1975.

Remark (Beeson 1972): E-HA$^\omega$ + AC refutes Church's thesis in the form

$$\forall f \exists e \forall x (fx = \{e\}(x))$$

(exercise! Hint: HA proves that there is no index e such that $\{e\}$ decides the Halting problem).

One might expect that E-HA$^\omega$ + AC refutes Church's thesis in the form

$$\forall x \exists y A(x, y) \to \exists e \forall x A(x, \{e\}(x)),$$

where A is a formula of HA (which is consistent with HA by Kleene realizability).

Proofs of extensional Goodman's theorem

First proof: Michael Beeson, *Goodman's theorem and beyond*, **Pacific J. Math.**, 1979.

More proofs: Lev Gordeev (1988) and, more recently, Benno van den Berg and Lotte van Slotte (2017).

One more proof: Emanuele Frittaion, *On Goodman realizability*, to appear in **NDJFL**. Blueprint: Goodman 1978. Different from Beeson's proof.

Finite-type arithmetic

Types ::= 0 (natural numbers) | $A \to B$ (functions) | $A \times B$ (products)

Application *App* (function symbols at all sensible types).

$App(f, x)$ (denoted fx).

$$\begin{array}{ccc} A \to B & & \\ \downarrow & & \\ App(f, x) & \leftarrow & B \\ \uparrow & & \\ A & & \end{array}$$

Finite-type arithmetic

NB: equality in all types.

- Axioms for Successor $0 \neq Sx$ and $Sx = Sy \rightarrow x = y$ at type 0
- Axioms for Combinators $\Pi xy = x$ and $\Sigma xyz = (xz)(yz)$ at all sensible types
- Axioms for Recursors $Rxy0 = x$ and $Rxy(Sz) = y(Rxyz)z$
- Induction
- $x = x$ at all types
- Decidable equality $x = y \vee x \neq y$ at type 0 (without extensionality we can have decidability at every type; extensionality plus decidable equality in all types gives excluded middle)
- Leibniz $x = y \wedge \varphi(x) \rightarrow \varphi(y)$

Finite-type arithmetic

- Extensionality $\forall x(fx = gx) \to f = g$

$$\therefore$$

- Axiom of choice AC

  $\forall x \exists y \varphi(x, y) \to \exists f \forall x \varphi(x, fx)$
- Axiom of relativized dependent choice RDC

  $\forall x(\varphi(x) \to \exists y(\varphi(y) \wedge \psi(x, y))) \to \forall x(\varphi(x) \to \exists f(f0 = x \wedge \forall n \psi(fn, f(Sn))))$
- Axiom of dependent choice DC

  $\forall x \exists y \psi(x, y) \to \forall x \exists f(f0 = x \wedge \forall n \psi(fn, f(Sn)))$

Goodman realizability, 1978

- If HA^ω proves φ then HA proves φ^{HRO}
- AC^{HRO} is false, although HA proves QF-AC^{HRO} (Troesltra, 1973). However, HA proves $a\colon AC^{HRO}$, where $a\colon \varphi$ is Kleene realizability with numbers
- If $HA^\omega + AC + RDC$ proves φ then HA proves $a\colon \varphi^{HRO}$ for some number a (soundness)
- Self-realizability does not work. A first-order formula φ is self-realizable if φ is equivalent to $\exists a(a\colon \varphi)$ provably in HA
- Goodman's solution: combine Kleene and Kripke
- Define $p \Vdash a\colon \varphi$, where p is a partial function from $\mathbb{N}$ to $\mathbb{N}$, a is a number, and φ is a sentence of HA^ω

Goodman realizability, 1978

1) For any definable set T of finite partial functions,

- if $\mathrm{HA}^\omega + \mathrm{AC} + \mathrm{RDC}$ proves φ then HA proves $\forall p \in T(p \Vdash a\colon \varphi)$, for some number a

2) Given a first-order sentence φ we can arithmetically define a set of finite partial functions T such that HA proves

- $\exists p(p \in T)$
- true-to-real: $\varphi \to \forall p \in T \exists a \exists q \in T(q \supseteq p \wedge q \Vdash a\colon \varphi)$
- real-to-true: $\forall p \in T \forall a((p \Vdash a\colon \varphi) \to \varphi)$

Goodman realizability, 1978

- $p \Vdash a\colon \alpha =_A \beta$ iff $|\alpha|_p$ and $|\beta|_p$ are both defined and $|\alpha|_p = |\beta|_p$
  ($|\alpha|_p$ is defined by recursion: $|\alpha\beta|_p \simeq \{|\alpha|_p\}^p(|\beta|_p)$)
- $p \Vdash a\colon \varphi \wedge \psi$ iff $p \Vdash (a)_0\colon \varphi$ and $p \Vdash (a)_1\colon \psi$
- $p \Vdash a\colon \varphi \vee \psi$ iff $(a)_0 = 0$ and $p \Vdash (a)_1\colon \varphi$ or else $(a)_0 = 1$ and $p \Vdash (a)_1\colon \psi$
- $p \Vdash a\colon \varphi \to \psi$ if for every $q \supseteq p$ and for every number b such that $q \Vdash b\colon \varphi$, there exists $r \supseteq q$ such that $\{a\}^r(b)$ is defined and $r \Vdash \{a\}^r(b)\colon \psi$
- $p \Vdash a\colon \exists x^A \varphi$ iff $p \Vdash (a)_0 \in A$ and $p \Vdash (a)_1\colon \varphi((a)_0)$
- $p \Vdash a\colon \forall x^A \varphi$ iff for every $q \supseteq p$ and for every number n such that $q \Vdash n \in A$, there exists $r \supseteq q$ such that $\{a\}^r(n)$ is defined and $r \Vdash \{a\}^r(n)\colon \varphi(n)$

Choice and extensionality

Goodman's Theorem (extensional)

E-HA$^\omega$ + AC + RDC is conservative over HA.

- HRO does not validate extensionality. Replace HRO with the model HEO of the hereditarily effective operations
- If E-HA$^\omega$ proves φ then HA proves φ^{HEO}
- AC$^{\mathsf{HEO}}$ is false.
- AC$^{\mathsf{HEO}}$ is not realizable (by Kleene recursive realizability). Already the HEO interpretation of AC^{1,0} is not Kleene realizable (hint: Halting problem).

What if

- Consider Kreisel modified realizability followed by HEO. This cannot work because independence of premise for ∃-free formulas is realizable
- Interpret E-HA$^\omega$ into HA by using HEO and use a version of Kleene realizability where realizers are elements of HEO. In particular, realizers of a formula φ have a type that depends on φ. This works for the soundness but then we have problems with self-realizability
- (Beeson's solution) Use a version of modified realizability for finite-type partial functionals followed by a version of HEO for finite-type partial functionals

Solution à la Goodman

My solution is to combine HEO with an extensional version of Kleene realizability.

We use HEO to interpret the quantifiers $\exists x^A$ and $\forall x^A$, but realizers are not exactly elements of HEO.

Actually, every formula is a type, the type of its realizers, and we have an extensional equality between realizers of the same formula.

On the other hand, extensionality is trivially realizable since it follows from its HEO interpretation.

Define $p \Vdash (a, b) \colon \varphi$, where p is a partial function from $\mathbb{N}$ to $\mathbb{N}$, a and b are natural numbers, and φ is a sentence of HA^ω

Choice and extensionality

Let us focus on the realizability part.

- $(a, b)\colon \alpha =_A \beta$ iff $|\alpha| =_A |\beta|$.
- $(a, b)\colon \varphi \wedge \psi$ iff $((a)_0, (b)_0)\colon \varphi$ and $((a)_1, (b)_1)\colon \psi$
- $(a, b)\colon \varphi \vee \psi$ iff either $(a)_0 = (b)_0 = 0$ and $((a)_1, (b)_1)\colon \varphi$ or else $(a)_0 = (b)_0 = 1$ and $((a)_1, (b)_1)\colon \psi$
- $(a, b)\colon \varphi \to \psi$ iff $(c, d)\colon \varphi$ implies $(\{a\}(c), \{b\}(d))\colon \psi$
- $(a, b)\colon \exists x^A \varphi$ iff $(a)_0 =_A (b)_0$ and $((a)_1, (b)_1)\colon \varphi((a)_0)$
- $(a, b)\colon \forall x^A \varphi$ iff $n =_A m$ implies $(\{a\}(n), \{b\}(m))\colon \varphi(n)$

Question

What is the second-order part of $\mathsf{HA}^\omega + \mathsf{AC}$? The second-order part of $\mathsf{HA}^\omega + \mathsf{QF\text{-}AC}$ is its restriction to second-order formulas.

$\therefore$

Generalization of Goodman realizability to higher types.

Question

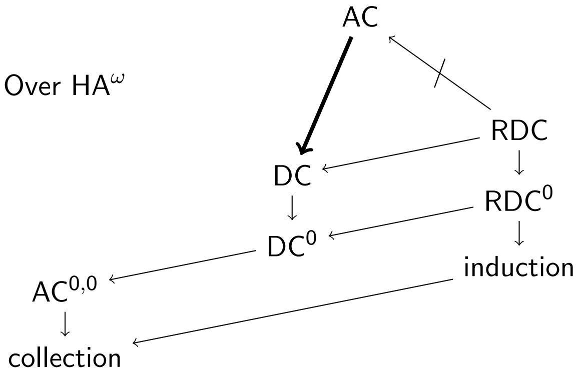
Goodman realizability validates both AC and RDC. What is the relation between these two choice principles?

$\therefore$

Over Zermelo-Fraenkel set theory ZF, AC $\rightarrow$ DC $\rightarrow$ AC^0. The implications are strict. AC^0 is countable axiom of choice.

The same implications hold over HA^ω.

DC $\nrightarrow$ AC over PA^ω (indeed $QF\text{-}AC^{1,0}$, which is provable in ZF).
And so neither does RDC.
(Ulrich Kohlenbach, *Remarks on Herbrand normal forms and Herbrand realizations*. **Archive for Mathematical Logic**, 1992)



Countable choice is to relativized dependent choice as collection is to induction.

The implications are quantifier-free induction.

The relation between AC and RDC has been open for at least 50 years. See:

William Howard and Georg Kreisel. *Transfinite induction and bar induction of types zero and one, and the role of continuity in intuitionistic analysis*. **J. Symbolic Logic**, 1966.

Georg Kreisel and Anne S Troelstra. *Formal systems for some branches of intuitionistic analysis*. **Annals of mathematical logic**, 1970.

Nicolas Goodman and John Myhill. *The formalization of Bishops constructive mathematics*. In **Toposes, Algebraic Geometry and Logic**, 1972.

Thanks for your attention!