Reflection ranks and proof theoretic ordinals

(based on joint work with James Walsh)

Fedor Pakhomov
Steklov Mathematical Institute, Moscow
pakhf@mi.ras.ru

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$\prec_{\mathsf{Con}}$-Order

$$T \prec_{\mathsf{Con}} U \overset{\text{def}}{\iff} U \text{ proves consistence of } T.$$

Empirical fact: $\prec_{\mathsf{Con}}$ is a linear well-founded preorder on natural theories

$$\mathsf{I}\Sigma_1 \prec_{\mathsf{Con}} \ldots \prec_{\mathsf{Con}} \mathsf{I}\Sigma_n \prec_{\mathsf{Con}} \mathsf{PA} \equiv_{\mathsf{Con}} \mathsf{ACA}_0$$

$$\mathsf{ACA}_0 \prec_{\mathsf{Con}} \Pi^1_1\text{-}\mathsf{CA}_0 \prec_{\mathsf{Con}} \Pi^1_2\text{-}\mathsf{CA}_0 \prec_{\mathsf{Con}} \ldots \prec_{\mathsf{Con}} \Pi^1_\infty\text{-}\mathsf{CA}_0 = \mathsf{PA}_2$$

$$\mathsf{PA}_2 \prec_{\mathsf{Con}} \mathsf{PA}_3 \prec_{\mathsf{Con}} \ldots \prec_{\mathsf{Con}} \mathsf{PA}_\infty \equiv_{\mathsf{Con}} \mathsf{Z}$$

$$\mathsf{Z} \prec_{\mathsf{Con}} \mathsf{Z}+\Delta_0\text{-}\mathsf{Coll} \prec_{\mathsf{Con}} \mathsf{Z}+\Pi_1\text{-}\mathsf{Coll} \prec_{\mathsf{Con}} \ldots \prec_{\mathsf{Con}} \mathsf{Z}+\Pi_\infty\text{-}\mathsf{Coll} = \mathsf{ZF}$$

$$\mathsf{ZF} \prec_{\mathsf{Con}} \mathsf{ZFC} + \exists\kappa\ \kappa \text{ is inaccessible} \prec_{\mathsf{Con}} \ldots$$

Although it is possible to construct artificial examples of descending chains consisting of true theories.

$$T_0 \succ_{\mathsf{Con}} T_1 \succ_{\mathsf{Con}} T_2 \succ_{\mathsf{Con}} \ldots$$

Π^1_1 soundndess and Π^1_1 reflection

Let T be an r.e. extension of ACA_0.

ACA_0 $=$PA $+$ second order axiom of induction$+$

$\exists X \forall x\, (\varphi(n) \leftrightarrow x \in X)$, for all arithmetical ($\Pi^0_\infty$) formulas $\varphi(x)$.

The Π^1_1 reflection principle $\mathsf{RFN}_{\Pi^1_1}(T)$ is Π^1_1 sentence expressing

T is Π^1_1-sound, e.g. T proves only true Π^1_1 sentences.

More formally $\mathsf{RFN}_{\Pi^1_1}(T)$ is given by the sentence

$$\forall \varphi \in \Pi^1_1\, (\mathsf{Prv}(T,\varphi) \to \mathsf{Tr}_{\Pi^1_1}(\varphi)),$$

where $\mathsf{Tr}_{\Pi^1_1}(x)$ is the partial truth definition for Π^1_1 formulas.

Well-foundedness in reflection order

We put

$$T \prec_{\Pi^1_1} U \overset{\text{def}}{\iff} U \vdash \mathsf{RFN}_{\Pi^1_1}(T).$$

Note that

$$T \prec_{\Pi^1_1} U \Rightarrow T \prec_{\mathsf{Con}} U.$$

Theorem

The restriction of $\prec_{\Pi^1_1}$ *on* Π^1_1*-sound extensions of* ACA_0 *is a well-founded relation.*

Proof of Well-Foundedness of $\prec_{\Pi^1_1}$

The negation of our theorem is the sentence DS

DS: "there is a descending chain in $\prec_{\Pi^1_1}$ starting with Π^1_1-sound r.e. theory"

We will show that $\mathsf{ACA}_0 + \mathsf{DS} \vdash \mathsf{Con}(\mathsf{ACA}_0 + \mathsf{DS})$. Then by Gödel's second incompleteness theorem $\mathsf{ACA}_0 + \mathsf{DS}$ is inconsistent and hence $\mathsf{ACA}_0 \vdash \neg\mathsf{DS}$.

Let us reason in $\mathsf{ACA}_0 + \mathsf{DS}$. We have sequence

$$T_0 \succ_{\Pi^1_1} T_1 \succ_{\Pi^1_1} \ldots,$$

where T_0 is Π^1_1-sound. Let S be the Σ^1_1-sentence saying that "there is a descending sequence in $\prec_{\Pi^1_1}$ starting from T_1." Since S is true and T_0 is Π^1_1-sound, there is a (countably coded) model

$$\mathfrak{M} \models T_0 + \mathsf{S}$$

But since T_0 proves Π^1_1-soundness of T_1,

$$\mathfrak{M} \models \mathsf{DS}.$$

The case of RCA_0

Over RCA_0 there are no truth definition for the class Π^1_1 but there are truth definitions for smaller classes $\Pi^1_1(\Pi^0_n)$, e.g. formulas of the form $\forall \vec{X}\ \varphi$, where $\varphi \in \Pi^0_n$. And we have reflection principles $\mathsf{RFN}_{\Pi^1_1(\Pi^0_n)}(T)$.

Theorem

The restriction of $\prec_{\Pi^1_1(\Pi^0_3)}$ *on* $\Pi^1_1(\Pi^0_3)$*-sound extensions of* RCA_0 *is a well-founded relation.*

Clarification: *Note that we need partial truth definition for class of formulas* Γ *to make reflection principle* RFN_Γ *a single sentence. Otherwise we put* RFN_Γ *be the scheme*

$$\forall \vec{x}\ (\mathsf{Prv}(T, \varphi(\vec{x})) \to \varphi(\vec{x})),\ \textit{where}\ \varphi \in \Gamma.$$

Reflection in first-order arithmetic

Over the system of first-order arithmetic EA we have partial truth definitions $\mathrm{Tr}_{\Pi^0_n}(x)$ and reflection principles $\mathrm{RFN}_{\Pi^0_n}(T)$.

Theorem (Friedman, Smorynski, Solovay)

There are no recursive sequences of theories $\langle T_i \mid i \in \mathbb{N} \rangle$ such that T_0 is consistent and

$$\mathrm{EA} \vdash \forall x \, \mathrm{Prv}(T_x, \ulcorner \mathrm{Con}(T_{\underline{x+1}}) \urcorner).$$

Theorem

There are no recursive sequences of theories $\langle T_i \mid i \in \mathbb{N} \rangle$ such that T_0 is Π^0_3-sound and

$$T_0 \succ_{\Pi^0_3} T_1 \succ_{\Pi^0_3} \ldots$$

Recursive descending chains

Recursive descending chain in $\prec_{\Pi^0_2}$:

$$T_0 \succ_{\Pi^0_2} T_1 \succ_{\Pi^0_2} T_2 \succ_{\Pi^0_2} \dots$$

$$T_n : \mathrm{I}\Sigma_1 + \text{“ either } \mathrm{RFN}_{\Pi^0_2}(\mathrm{PA}) \text{ or } \mathrm{RFN}^{p-n}_{\Pi^0_2}(\mathrm{I}\Sigma_1) \text{, where } p \text{ is Gödel number of the first proof of false } \Sigma^0_1 \text{ sentence in PA”}$$

Note that all T_n are true arithmeical theories.

Reflection Rank

For an r.e. extension T of ACA_0 we put

$$|T|_{\mathsf{ACA}_0} = \alpha \text{ if } T \text{ is in well-founded part of } \prec_{\Pi^1_1} \text{ and } \alpha \text{ is it's well-founded rank}$$

$$|T|_{\mathsf{ACA}_0} = \infty, \text{ otherwise}$$

More standard measure is Π^1_1 proof-theoretic ordinal:

$$|T|_{\mathsf{WO}} = \sup\{|\alpha| \mid \alpha \text{ is recursive linear order and } T \vdash \mathsf{WO}(\alpha)\}.$$

Reflection ranks and proof-theoretic ordinals of some theories:

	$\lvert\cdot\rvert_{\mathsf{ACA}_0}$	$\lvert\cdot\rvert_{\mathsf{WO}}$
ACA_0	0	ε_0
$\mathsf{ACA}_0 + \mathsf{Con}(\mathsf{ACA}_0)$	0	ε_0
$\mathsf{ACA}_0 + \mathsf{RFN}_{\Pi^1_1}(\mathsf{ACA}_0)$	1	ε_1
ACA_0'	ω	ε_ω
ACA	ε_0	$\varepsilon_{\varepsilon_0}$
ACA_0^+	$\varphi(2,0)$	$\varphi(2,0)$
ATR_0	Γ_0	Γ_0

Iterations of reflection principles

For recursive ordinal notations α we could define iterations $\mathsf{RFN}^{\alpha}_{\Gamma}(T)$:

- $\mathsf{RFN}^{0}_{\Gamma}(T) = T$
- $\mathsf{RFN}^{\alpha+1}_{\Gamma}(T) = T + \mathsf{RFN}_{\Gamma}(\mathsf{RFN}^{\alpha}_{\Gamma}(T))$
- $\mathsf{RFN}^{\lambda}_{\Gamma}(T) = \bigcup\limits_{\alpha<\lambda} \mathsf{RFN}^{\alpha}_{\Gamma}(T)$, $\lambda \in \mathbf{Lim}$.

Theorem (Turing)

For each true Π_1 *sentence* F *there is recursive ordinal notation* α

$$\mathsf{Con}^{\alpha}(\mathsf{PA}) \vdash \mathsf{F}.$$

Theorem (Feferman)

For each true Π^0_∞ *sentence* F *there is recursive ordinal notation* α

$$\mathsf{RFN}^{\alpha}_{\Pi^0_\infty}(\mathsf{PA}) \vdash \mathsf{F}.$$

Iterations of Π^1_1-reflection

Theorem

$$\mathsf{RFN}^{\alpha}_{\Pi^1_1}(\mathsf{ACA}_0) \equiv_{\Pi^1_1(\Pi^0_3)} \mathsf{RFN}^{\varepsilon_\alpha}_{\Pi^1_1(\Pi^0_3)}(\mathsf{RCA}_0)$$

Proposition

$|\mathsf{RFN}^{\beta}_{\Pi^1_1(\Pi^0_3)}(\mathsf{RCA}_0)|_{\mathsf{RCA}_0} = |\beta|$

Proposition

$$\mathsf{ACA}_0 \vdash \forall \alpha \, (\mathsf{WO}(\alpha) \leftrightarrow \mathsf{RFN}^{\alpha+1}_{\Pi^1_1(\Pi^0_3)}(\mathsf{RCA}_0))$$

Corollary

$$|\mathsf{RFN}^{\alpha}_{\Pi^1_1}(\mathsf{ACA}_0)|_{\mathsf{WO}} = |\varepsilon_\alpha|.$$

Proving $\mathsf{RFN}^{\alpha}_{\Pi^1_1}(\mathsf{ACA}_0) \equiv_{\Pi^1_1(\Pi^0_3)} \mathsf{RFN}^{\varepsilon_\alpha}_{\Pi^1_1(\Pi^0_3)}(\mathsf{RCA}_0)$

Let us consider pseudo-Π^1_1 language $\mathbf{\Pi}^0_\infty$, i.e. arithmetical formulas $\varphi(X)$ with free unary predicate X. We have embedding of pseudo-Π^1_1 language into second-order arithmetic $\varphi(X) \longmapsto \forall X\ \varphi(X)$.

$$\mathsf{RFN}^{\alpha}_{\Pi^1_1}(\mathsf{ACA}_0) \equiv_{\mathbf{\Pi}^0_\infty} \mathsf{RFN}^{\alpha}_{\mathbf{\Pi}^0_\infty}(\mathsf{PA}(X)),$$

$$\mathsf{RFN}^{\alpha}_{\Pi^1_1(\Pi^0_3)}(\mathsf{RCA}_0) \equiv_{\mathbf{\Pi}^0_3} \mathsf{RFN}^{\alpha}_{\mathbf{\Pi}^0_3}(\mathsf{I\Sigma}_1(X)).$$

Schmerl-style formula for uniform pseudo-Π^1_1 reflection

$$\mathsf{RFN}^{\alpha}_{\mathbf{\Pi}^0_\infty}(\mathsf{PA}(X)) \equiv_{\mathbf{\Pi}^0_3} \mathsf{RFN}^{\varepsilon_\alpha}_{\mathbf{\Pi}^0_3}(\mathsf{I\Sigma}_1)$$

Thus

$$\mathsf{RFN}^{\alpha}_{\Pi^1_1}(\mathsf{ACA}_0) \equiv_{\mathbf{\Pi}^0_\infty} \mathsf{RFN}^{\alpha}_{\mathbf{\Pi}^0_\infty}(\mathsf{PA}(X)) \equiv_{\mathbf{\Pi}^0_3} \mathsf{RFN}^{\varepsilon_\alpha}_{\mathbf{\Pi}^0_3}(\mathsf{I\Sigma}_1) \equiv_{\mathbf{\Pi}^0_3} \mathsf{RFN}^{\varepsilon_\alpha}_{\Pi^1_1(\Pi^0_3)}(\mathsf{RCA}_0)$$

Calculus RC_0

Beklemishev approach to proof of Schmerl formula employs ordinal notation system based on reflection principles.

Reflection calculus RC:
Formulas:

$$F ::= \top \mid F \wedge F \mid \Diamond_n F, \text{ where } n \text{ ranges over } \mathbb{N}.$$

Sequents:

$$A \vdash B, \text{ for RC-formulas } A \text{ and } B.$$

1. $A \vdash A$; $A \vdash \top$; if $A \vdash B$ and $B \vdash C$ then $A \vdash C$;
2. $A \wedge B \vdash A$; $A \wedge B \vdash B$; if $A \vdash B$ and $A \vdash C$ then $A \vdash B \wedge C$;
3. if $A \vdash B$ then $\Diamond_n A \vdash \Diamond_n B$, for all $n \in \mathbb{N}$;
4. $\Diamond_n \Diamond_n A \vdash \Diamond_n A$, for every $n \in \mathbb{N}$;
5. $\Diamond_n A \vdash \Diamond_m A$, for all $n > m$;
6. $\Diamond_n A \wedge \Diamond_m B \vdash \Diamond_n (A \wedge \Diamond_m B)$, for all $n > m$.

Beklemishev's Ordinal Notation System

$$A <_0 B \overset{\text{def}}{\Longleftrightarrow} B \vdash \Diamond_0 A$$

$$A \sim B \overset{\text{def}}{\Longleftrightarrow} A \vdash B \text{ and } B \vdash A$$

Theorem (Beklemishev)

$(\mathsf{RC}_0/{\sim}, <_0)$ *is a well-ordering with order type* ε_0.

It were done by Beklemishev by embedding this system in Cantor ordinal notation system for ε_0.

Well-Foundedness Proof

Let us interpret RC-formulas by $\mathcal{L}_2$-theories. We interpret $\top$ as $\top^\star = \mathsf{ACA}_0$. And we interpret $\Diamond_n A$ as $(\Diamond_n A)^\star = \mathsf{RFN}_{\Pi^1_{n+1}}(A^\star)$.

It is easy to see that $A \vdash B$ implies $A^\star \vdash B^\star$.
Hence $A <_0 B$ implies $A^\star <_{\Pi^1_1} B^\star$.
Thus $<_0$ is a well-founded relation on the set of RC_0 formulas.

Thank You!