The bi-embeddability relation for countable abelian group

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joint work with Simon Thomas

Logic Colloquium 2018

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- X_{TA} Polish space of countable torsion abelian groups.
- $\cong_{\mathsf{TA}}$ isomorphism on X_{TA}.
- $\equiv_{\mathsf{TA}}$ bi-embeddability relation on X_{TA}.

We analyze the Borel complexity of $\cong_{\mathsf{TA}}$ and $\equiv_{\mathsf{TA}}$ within the class of analytic equivalence relation.

Theorem (Folklore)

For every $A \in X_{\mathsf{TA}}$

$$A = \bigoplus_{p \text{ prime}} A_p$$

where $A_p = \{a \in A \mid o(a) = p^n \text{ for some } n \in \mathbb{N}\}$.

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Fact

For every $A, B \in X_{\mathsf{TA}}$,

- $A \cong_{\mathsf{TA}} B \iff A_p \cong_p B_p$, *for all prime p;*
- $A \equiv_{\mathsf{TA}} B \iff A_p \equiv_p B_p$, *for all prime p.*

Ulm Theory in a nutshell

Let $A \in X_p$. The **Ulm subgroups** of A are

$$\begin{cases} A^0 = A, \\ A^{\alpha+1} = \bigcap_{n \in \mathbb{N}} p^n A^\alpha, \\ A^\lambda = \bigcap_{\alpha < \lambda} A^\alpha, & \text{for } \lambda \text{ limit.} \end{cases}$$

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- **Rank of (the divisible part of)** A.
$A^{\tau(A)} = \underbrace{\mathbb{Z}(p^\infty) \oplus \cdots \oplus \mathbb{Z}(p^\infty) \oplus \dots}_{d \text{ times}} = \mathbb{Z}(p^\infty)^{(d)}$, with $d \leq \omega$.

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1. **Σ-cyclic groups**. Let C_k denote the cyclic group of order k. If $A = \bigoplus_{n \in \mathbb{N}} C_{p^n}$, then $A^1 = 0$.

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2. Let $G = \bigoplus_{i\in\mathbb{N}} C_{p^{\ell+\ell_i}}$, with (ℓ_i) unbounded, and $C_{p^{\ell+\ell_i}} = \langle c_i \rangle$.

$$B := G/H,$$

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with $H = \langle p^{\ell_i} c_i - p^{\ell_j} c_j \mid i, j \in \mathbb{N} \rangle$. Then

$$\tau(B) = 2 \quad \text{and} \quad \begin{cases} B_0 \cong \bigoplus_{i\in\mathbb{N}} C_{p^{\ell_i}}, \\ B_1 \cong C_{p^\ell}. \end{cases}$$

Lemma (C.-Thomas; aprés Barwise-Eklof, 1970)

Given $A, B \in X_p$, we have $A \equiv_p B$ if and only if either

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2. $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)}) < \omega$ *and*
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Corollary

The are exactly ω_1 countable abelian p-groups up to $\equiv_p$. Thus, bi-embeddability relation $\equiv_p$ is not Borel.

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- For all $\alpha < \tau(A)$, the Ulm factor A_α is an unbounded Σ-cyclic.
- Any two unbounded Σ-cyclic p-groups are bi-embeddable.
- There are countably many Σ-cyclic p-groups up to $\equiv_p$. □

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$$\mathrm{Tr} \to X_p, \quad T \mapsto A_T := \langle t \in T \mid \emptyset = 0, pt = t^- \rangle.$$

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   So, the map $T \mapsto A_T^{(\omega)}$ reduces ill-founded trees to C_∞. $\square$

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  If $\mathbb{P}$ is the Levy collapse $\mathrm{Col}(\omega_1, \omega)$, then the number of pinned $\mathbb{P}$-names for $\cong_{\mathsf{TA}}$ is consistently bigger then the number of pinned $\mathbb{P}$-names for $\equiv_{\mathsf{TA}}$. Then, argue by absoluteness.

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  In fact, we show that $\equiv_p \not\leq_B \cong_{\mathsf{TA}}$, for every prime p. $\square$

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See the blackboard!

Torsion-free abelian groups

Theorem (C.-Thomas)

The bi-embeddability relation $\equiv_{\mathsf{TFA}}$ for countable torsion-free abelian groups is a complete $\boldsymbol{\Sigma}^1_1$ equivalence relation.

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Thus $\equiv_{\mathsf{TFA}}$ is strictly above $\cong_{\mathsf{TFA}}$, that is known to be:

- a complete $\mathbf{\Sigma}^1_1$ subset of $X_{\mathsf{TFA}} \times X_{\mathsf{TFA}}$;

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- possibly complete for the S_∞-orbit equivalence relations.

  (Friedman-Stanley conjecture)

Beyond Borel reducibility

$\equiv_\mathrm{p}$ and $\cong_\mathrm{p}$ are incomparable but...

...we can select a $\cong_\mathrm{p}$-class within every $\equiv_\mathrm{p}$-class.

Theorem (C.-Thomas)

There is a function $\theta\colon X_p \to X_p$ such that

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Theorem (C.-Thomas)

There is a $\mathbf{\Delta}^1_2$ function $\theta\colon X_p \to X_p$ such that

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- *$\theta(A) \equiv_p A$, for all $A \in X_p$.*

Thus, we have $\equiv_p \leq_{\mathbf{\Delta}^1_2} \cong_p$ and $\equiv_{\mathrm{TA}} \leq_{\mathbf{\Delta}^1_2} \cong_{\mathrm{TA}}$.

Simpler than isomorphism

Theorem (C.-Thomas)

If a Ramsey cardinal exists, then

- $\cong_p \not\leq_{\mathbf{\Delta}^1_2} \equiv_p$, *for every prime p;*
- $\cong_{\mathsf{TA}} \not\leq_{\mathbf{\Delta}^1_2} \equiv_{\mathsf{TA}}$.

In fact, if Projective Determinacy holds then there is not any $\mathbf{\Delta}^1_{<\omega}$ reduction from $\cong_p$ to $\equiv_p$ (from $\cong_{\mathsf{TA}}$ to $\equiv_{\mathsf{TA}}$).

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Thank you!