Filippo Cavallari

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Descriptive Set Theory and Automata

27 July 2018

Field of research

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Automata Theory

Descriptive Set Theory

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Automata Theory Descriptive Set Theory

$\longrightarrow$

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Automata Theory $\qquad\qquad$ Descriptive Set Theory

$\downarrow \qquad \longrightarrow \qquad \downarrow$

A^ω, Tr_A $\qquad\qquad$ Polish spaces

Infinite words

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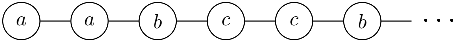

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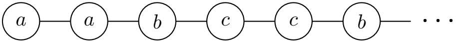

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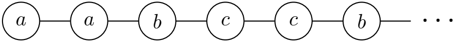

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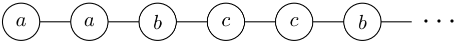

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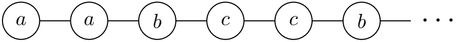

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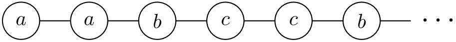

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Regular word languages: parity automata, MSO, regular expressions, ...

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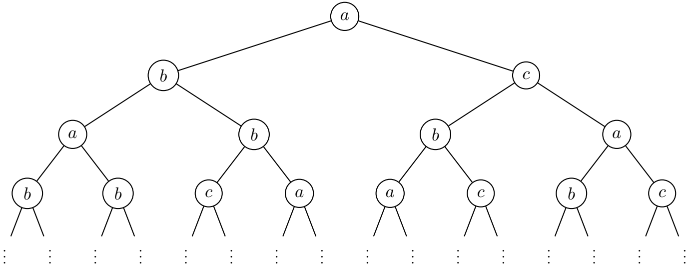

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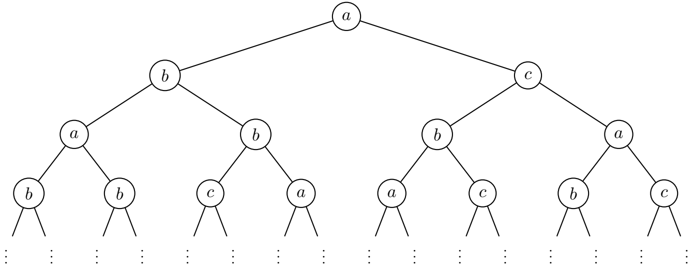

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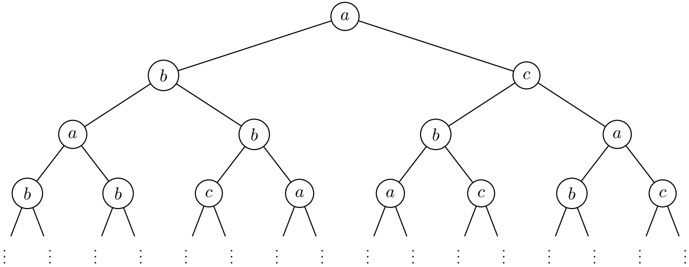

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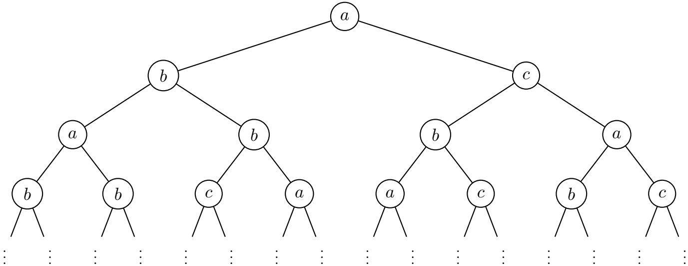

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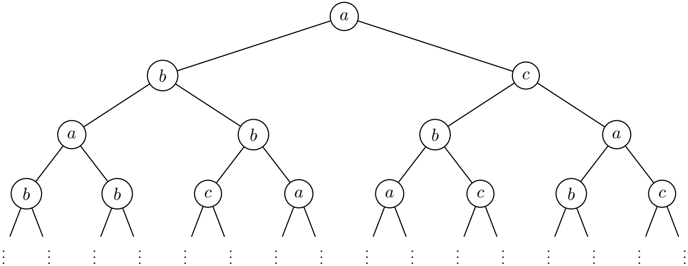

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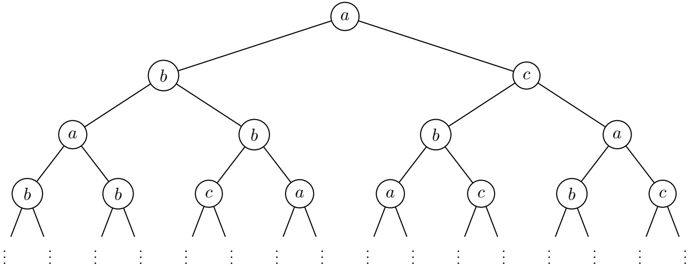

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A^ω and Tr_A are Polish spaces (homeomorphic to the Cantor space 2^ω).

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Theorem

The following properties hold:

• Word regular languages have Wadge degrees with Wadge rank of the form:

$$\omega_1^k n_k + \ldots + \omega_1 n_1 + n_0,$$

where $n_0, \ldots, n_k, k \in \omega$.

• Languages recognised by word parity deterministic automata that use k priorities correspond to Wadge degrees with Wadge rank less than ω_1^k.

• Given a regular language L, we have an effective algorithm that establishes its precise position in the Wadge hierarchy.

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We instead focus on GENERAL regular tree languages, considering the first two levels of the Borel hierarchy.

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The membership problem for each of the class $\mathbf{\Delta}_1^0, \mathbf{\Sigma}_1^0, \mathbf{\Pi}_1^0$ is decidable.

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Thank you!