Absorbing the Structural Rules in the Sequent Calculus with Additional Atomic Rules

Franco Parlamento University of Udine
Flavio Previale University of Turin

Logic Colloquium 2018 Udine July 27th 2018

The **G3[mic]** systems

Initial sequents

$$P, \Gamma \Rightarrow \Delta, P \qquad\qquad (\bot, \Gamma \Rightarrow \Delta, \bot \quad \text{in } \mathbf{G3m})$$

The **G3[mic]** systems

Initial sequents

$$P, \Gamma \Rightarrow \Delta, P \qquad (\perp, \Gamma \Rightarrow \Delta, \perp \text{ in } \mathbf{G3m})$$

Logical Rules

$$\frac{}{\perp, \Gamma \Rightarrow \Delta} \qquad (\text{ not in } \mathbf{G3m})$$

The **G3[mic]** systems

Initial sequents

$$P, \Gamma \Rightarrow \Delta, P \qquad (\bot, \Gamma \Rightarrow \Delta, \bot \ \text{ in } \mathbf{G3m})$$

Logical Rules

$$\overline{\bot, \Gamma \Rightarrow \Delta} \qquad (\text{ not in } \mathbf{G3m})$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

The **G3[mic]** systems

Initial sequents

$$P, \Gamma \Rightarrow \Delta, P \qquad (\bot, \Gamma \Rightarrow \Delta, \bot \text{ in } \mathbf{G3m})$$

Logical Rules

$$\overline{\bot, \Gamma \Rightarrow \Delta} \qquad (\text{ not in } \mathbf{G3m})$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\ L\rightarrow \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ R\rightarrow$$

$$\frac{A[x/t], \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} \; L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A[x/a]}{\Gamma \Rightarrow \Delta, \forall x A} \; R\forall$$

$$\frac{A[x/t], \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} \; L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A[x/a]}{\Gamma \Rightarrow \Delta, \forall x A} \; R\forall$$

$$\frac{A[x/a], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta} \; L\exists \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A} \; R\exists$$

In **G3mi** the rules $L\rightarrow$, $R\rightarrow$ and $R\forall$ are replaced by:

$$\frac{A \rightarrow B, \Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \; L^i \rightarrow \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \; R^i \rightarrow$$

$$\frac{\Gamma \Rightarrow A[x/a]}{\Gamma \Rightarrow \Delta, \forall x A} \; R^i \forall$$

GK3[mic]

GK3[mic]

$$\frac{A \wedge B, A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

Atomic Rules

$$\frac{\vec{Q}_1, \Gamma_1 \Rightarrow \Delta_1, \vec{Q}'_1 \quad \ldots \quad \vec{Q}_n, \Gamma_n \Rightarrow \Delta_n, \vec{Q}'_n}{\vec{P}, \Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_n, \vec{P'}}$$

where $\vec{Q}_1, \vec{Q}'_1, \ldots, \vec{Q}_n, \vec{Q}'_n, \vec{P}, \vec{P'}$ are sequences (possibly empty) of atomic formulae and $\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots \Delta_n$ are finite sequences (possibly empty) of formulae that are not active in t

Equality Rules

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Ref} \qquad \frac{}{\Gamma \Rightarrow \Delta, t = t}\ \overline{\text{Ref}}$$

Equality Rules

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Ref} \qquad \frac{}{\Gamma \Rightarrow \Delta, t = t}\ \overline{\text{Ref}}$$

$$\frac{s = r, P[x/s], P[x/r], \Gamma \Rightarrow \Delta}{s = r, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Repl}$$

Equality Rules

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Ref} \qquad \frac{}{\Gamma \Rightarrow \Delta, t = t}\ \overline{\text{Ref}}$$

$$\frac{s = r, P[x/s], P[x/r], \Gamma \Rightarrow \Delta}{s = r, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Repl}$$

$$\frac{r = s, P[x/r], \Gamma \Rightarrow \Delta}{r = s, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Repl}_1^l \qquad \frac{r = s, \Gamma \Rightarrow \Delta, P[x/r]}{r = s, \Gamma \Rightarrow \Delta, P[x/s]}\ \text{Repl}_1^r$$

Equality Rules

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Ref} \qquad \frac{}{\Gamma \Rightarrow \Delta, t = t}\ \overline{\text{Ref}}$$

$$\frac{s = r, P[x/s], P[x/r], \Gamma \Rightarrow \Delta}{s = r, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Repl}$$

$$\frac{r = s, P[x/r], \Gamma \Rightarrow \Delta}{r = s, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Repl}^l_1 \qquad \frac{r = s, \Gamma \Rightarrow \Delta, P[x/r]}{r = s, \Gamma \Rightarrow \Delta, P[x/s]}\ \text{Repl}^r_1$$

$$\frac{s = r, P[x/r], \Gamma \Rightarrow \Delta}{s = r, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Repl}^l_2 \qquad \frac{s = r, \Gamma \Rightarrow \Delta, P[x/r]}{s = r, \Gamma \Rightarrow \Delta, P[x/s]}\ \text{Repl}^r_2$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, r = s \qquad \Gamma_2 \Rightarrow \Delta_2, P[x/r]}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, P[x/s]} \ \text{CNG}$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, r = s \qquad \Gamma_2 \Rightarrow \Delta_2, P[x/r]}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, P[x/s]} \quad \text{CNG}$$

Admissibilty of the structural rules in **G3**[*mic*]

Admissibilty of the structural rules in **G3**[*mic*]

i) Hight preserving admissibility of the weakening rules

Admissibilty of the structural rules in **G3**[*mic*]

i) Hight preserving admissibility of the weakening rules

ii) Height preserving invertibility of the logical rules different from $R^i \rightarrow$ and $R^i\forall$

Admissibilty of the structural rules in **G3**[*mic*]

i) Hight preserving admissibility of the weakening rules

ii) Height preserving invertibility of the logical rules different from $R^i \rightarrow$ and $R^i\forall$

iii) Admissibility of the contraction rules **via hight preserving admissibility**

Admissibilty of the structural rules in **G3**[*mic*]

i) Hight preserving admissibility of the weakening rules

ii) Height preserving invertibility of the logical rules different from $R^i \rightarrow$ and $R^i\forall$

iii) Admissibility of the contraction rules **via hight preserving admissibility**

iv) Admissibility of the Cut rule, using *i*) and *iii*) “all the time"

$$a = f(a),\ a = f(a) \Rightarrow a = f(f(a))$$

has derivations of height equal 1 in the systems obtained by adding *Ref* and *Repl* to **G3**[**mic**]:

$$a = f(a),\ a = f(a) \Rightarrow a = f(f(a))$$

has derivations of height equal 1 in the systems obtained by adding *Ref* and *Repl* to **G3**[**mic**]:

$$\frac{a = f(a),\ a = f(a),\ a = f(f(a)) \Rightarrow a = f(f(a))}{a = f(a),\ a = f(a) \Rightarrow a = f(f(a))}$$

$$a = f(a),\ a = f(a) \Rightarrow a = f(f(a))$$

has derivations of height equal 1 in the systems obtained by adding *Ref* and *Repl* to **G3**[**mic**]:

$$\frac{a = f(a),\ a = f(a),\ a = f(f(a)) \Rightarrow a = f(f(a))}{a = f(a),\ a = f(a) \Rightarrow a = f(f(a))}$$

but $a = f(a) \Rightarrow a = f(f(a))$ cannot have a derivation of height less than or equal 1 in such a system.

The Separation Theorem

Definition

A derivation in **G3[mic]**$^{\mathcal{R}} + Cut_{cs}$ is said to be *separated* if no logical inference precedes an $\mathcal{R}$ or Cut_{cs}-inference.

The Separation Theorem

Definition

A derivation in $\mathbf{G3[mic]}^{\mathcal{R}} + Cut_{cs}$ is said to be *separated* if no logical inference precedes an $\mathcal{R}$ or Cut_{cs}-inference.

Theorem

Every derivation in $\mathbf{G3[mic]}^{\mathcal{R}} + Cut_{cs}$ *can be transformed into a separated derivation of its endsequent.*

The $\mathcal{R}$-Admissibility Theorem

Theorem

If the structural rules are admissible in $\mathcal{R}$, then they are admissible in $\mathbf{G3[mic]}^{\mathcal{R}}$ *as well.*

The $\mathcal{R}$-Admissibility Theorem

Theorem

If the structural rules are admissible in $\mathcal{R}$, then they are admissible in $\mathbf{G3[mic]}^{\mathcal{R}}$ *as well.*

$$\dfrac{\begin{array}{c}\mathcal{E} \\ F, F, \Gamma \Rightarrow \Delta\end{array}}{F, \Gamma \Rightarrow \Delta}$$

The $\mathcal{R}$-Admissibility Theorem

Theorem

If the structural rules are admissible in $\mathcal{R}$, then they are admissible in $\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$ *as well.*

$$\dfrac{\begin{array}{c}\mathcal{E}\\ F, F, \Gamma \Rightarrow \Delta\end{array}}{F, \Gamma \Rightarrow \Delta}$$

can be replaced by:

$$\dfrac{\begin{array}{c}\mathcal{I}\\ F, \Gamma \Rightarrow F\end{array} \qquad \begin{array}{c}\mathcal{E}\\ F, F, \Gamma \Rightarrow \Delta\end{array}}{F, \Gamma \Rightarrow \Delta}$$

where, in case F is not atomic, $\mathcal{I}$ is a derivation in **G3m** or in **G3i**.

Proof of the Separation Theorem

Lemma

If the premisses of an $\mathcal{R}$-inference R have a separated derivation, then its conclusion has a separated derivation.

Proof of the Separation Theorem

Lemma

If $\Gamma \Rightarrow \Delta, A$ *and* $A, \Gamma \Rightarrow \Delta$ *have separated derivation in* $\mathbf{G3c[mic]}^{\mathcal{R}} + Cut_{cs}$*, then* $\Gamma \Rightarrow \Delta$ *has a separated derivation in the same system.*

Proof of the Separation Theorem

Lemma

a) *Hight-preserving separated invertibility of the logical rules in* $\mathbf{G3c}^{\mathcal{R}} + Cut_{cs}$
If the conclusion of a logical rule has a separated derivation of height bounded by h, then also its premisses have separated derivations of height bounded by h.

b) *The same holds for* $\mathbf{G3[mi]}^{\mathcal{R}} + Cut_{cs}$*, except for the rules* $R^i \rightarrow$ *and* $R^i\forall$*.*

Proof of the Separation Theorem

Both $\mathcal{D}$ and $\mathcal{E}$ end with a logical rule:

Proof of the Separation Theorem

Both $\mathcal{D}$ and $\mathcal{E}$ end with a logical rule:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \Rightarrow \Delta, A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ A, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

Proof of the Separation Theorem

Both $\mathcal{D}$ and $\mathcal{E}$ end with a logical rule:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \Rightarrow \Delta, A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ A, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

Principal induction on $h(A)$, secondary induction on $h(\mathcal{D}) + h(\mathcal{E})$

Proof of the Separation Theorem

Classical case

- A not principal in the last inference of $\mathcal{D}$ or in the last inference of $\mathcal{E}$
- A principal in both the last inference of $\mathcal{D}$ and the last inference of $\mathcal{E}$

Proof of the Separation Theorem

Classical case

- A not principal in the last inference of $\mathcal{D}$ or in the last inference of $\mathcal{E}$
- A principal in both the last inference of $\mathcal{D}$ and the last inference of $\mathcal{E}$

Minimal and intuitionistic case

- A not principal in the last inference of $\mathcal{D}$
- A principal in the last inference of $\mathcal{D}$ but not in the last inference of $\mathcal{E}$
- A principal in both the last inference of $\mathcal{D}$ and the last inference of $\mathcal{E}$

Proof of the Separation Theorem

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \Delta', E \rightarrow F, \exists xB, B[x/t]\end{array}}{\Gamma \Rightarrow \Delta', E \rightarrow F, \exists xB} \qquad \dfrac{\begin{array}{c}\mathcal{E}_0\\ \exists xB, E, \Gamma \Rightarrow F\end{array}}{\exists xB, \Gamma \Rightarrow \Delta', E \rightarrow F}}{\Gamma \Rightarrow \Delta', E \rightarrow F}$$

Proof of the Separation Theorem

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \Delta', E \rightarrow F, \exists x B, B[x/t]\end{matrix}}{\Gamma \Rightarrow \Delta', E \rightarrow F, \exists x B} \qquad \dfrac{\begin{matrix}\mathcal{E}_0\\ \exists x B, E, \Gamma \Rightarrow F\end{matrix}}{\exists x B, \Gamma \Rightarrow \Delta', E \rightarrow F}}{\Gamma \Rightarrow \Delta', E \rightarrow F}$$

is transformed into:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \Delta', E \rightarrow F, \exists x B, B[x/t]\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{E}_0\\ \exists x B, E, \Gamma \Rightarrow F\end{matrix}}{\exists x B, \Gamma \Rightarrow \Delta', E \rightarrow F, B[x/t]}}{\Gamma \Rightarrow \Delta', E \rightarrow F, B[x/t]} \text{ ind}$$

Proof of the Separation Theorem

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \Delta', E \to F, \exists x B, B[x/t]\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{E}_0\\ \exists x B, E, \Gamma \Rightarrow F\end{array}}{\exists x B, \Gamma \Rightarrow \Delta', E \to F, B[x/t]}}{\Gamma \Rightarrow \Delta', E \to F, B[x/t]}\ \text{ind}$$

Proof of the Separation Theorem

$$\dfrac{\overset{\mathcal{D}_0}{\Gamma \Rightarrow \Delta', E \to F, \exists x B, B[x/t]} \qquad \dfrac{\overset{\mathcal{E}_0}{\exists x B, E, \Gamma \Rightarrow F}}{\exists x B, \Gamma \Rightarrow \Delta', E \to F, B[x/t]}}{\Gamma \Rightarrow \Delta', E \to F, B[x/t]}\ \text{ind}$$

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{E}_0}{\exists x B, E, \Gamma \Rightarrow F}}{\exists x B, \Gamma \Rightarrow \Delta', E \to F}}{B[x/a], \Gamma \Rightarrow \Delta', E \to F}\ \text{inv}}{B[x/t], \Gamma \Rightarrow \Delta', E \to F}\ \text{Sub}[a/t]$$

Proof of the Separation Theorem

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \Delta', E \to F, \exists xB, B[x/t]\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{E}_0\\ \exists xB, E, \Gamma \Rightarrow F\end{matrix}}{\exists xB, \Gamma \Rightarrow \Delta', E \to F, B[x/t]}}{\Gamma \Rightarrow \Delta', E \to F, B[x/t]}\ \text{ind}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{E}_0\\ \exists xB, E, \Gamma \Rightarrow F\end{matrix}}{\exists xB, \Gamma \Rightarrow \Delta', E \to F}}{B[x/a], \Gamma \Rightarrow \Delta', E \to F}\ \text{inv}}{B[x/t], \Gamma \Rightarrow \Delta', E \to F}\ \text{Sub}[a/t]$$

$$\dfrac{\Gamma \Rightarrow \Delta', E \to F, B[x/t] \qquad B[x/t], \Gamma \Rightarrow \Delta', E \to F}{\Gamma \Rightarrow \Delta', E \to F}\ \text{ind}$$

Logical applications

For $\mathcal{R} = \emptyset$:

Admissibility of the structural rules in G3[mic], without using height-preserving admissibility of the contraction rules

Logical applications

If the structural rules are admissible in $\mathcal{R}$, then they are admissible in $\mathbf{GK3}[\mathbf{mic}]^{\mathcal{R}}$ as well.

Logical applications

If the structural rules are admissible in $\mathcal{R}$, then they are admissible in $\mathbf{GK3[mic]}^{\mathcal{R}}$ as well.

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}\,\text{LW} \quad \dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,\text{LW}}{A, B \Rightarrow A \wedge B}\,R\wedge \qquad A \wedge B, A, B, \Gamma \Rightarrow \Delta}{\dfrac{\dfrac{A, B, A, B, \Gamma \Rightarrow \Delta}{A, B, \Gamma \Rightarrow \Delta}\,\text{LC}}{A \wedge B, \Gamma \Rightarrow \Delta}\,L\wedge}\,\text{Cut}$$

Applications to logic with equality

The systems that contain Ref or $\overline{\text{Ref}}$ and at least one of the equality rules for which the structural rules are admissible are all equivalent.

Applications to logic with equality

$\mathbf{G3[mic]}^{=} = \mathbf{G3[mic]} + \{\mathrm{Ref}, \mathrm{Repl}\}$

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{Ref} \qquad \frac{s = r,\ P[x/s],\ P[x/r],\ \Gamma \Rightarrow \Delta}{s = r,\ P[x/s],\ \Gamma \Rightarrow \Delta}\ \mathrm{Repl}$$

Applications to logic with equality

$\mathbf{G3[mic]}^{=} = \mathbf{G3[mic]} + \{\mathrm{Ref}, \mathrm{Repl}\}$

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{Ref} \qquad\qquad \frac{s = r,\ P[x/s],\ P[x/r],\ \Gamma \Rightarrow \Delta}{s = r,\ P[x/s],\ \Gamma \Rightarrow \Delta}\ \mathrm{Repl}$$

Theorem

The structural rules are admissible in $\mathbf{G3[mic]}^{=}$

Applications to logic with equality

$\mathbf{G3[mic]}^{=} = \mathbf{G3[mic]} + \{\mathrm{Ref}, \mathrm{Repl}\}$

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{Ref} \qquad \frac{s = r,\ P[x/s],\ P[x/r],\ \Gamma \Rightarrow \Delta}{s = r,\ P[x/s],\ \Gamma \Rightarrow \Delta}\ \mathrm{Repl}$$

Theorem

The structural rules are admissible in $\mathbf{G3[mic]}^{=}$

Applications to logic with equality

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Ref} \qquad \frac{s = r,\ P[x/r],\ \Gamma \Rightarrow \Delta}{s = r,\ P[x/s],\ \Gamma \Rightarrow \Delta}\ \text{Repl}_2^l$$

Applications to logic with equality

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Ref} \qquad \frac{s = r,\ P[x/r],\ \Gamma \Rightarrow \Delta}{s = r,\ P[x/s],\ \Gamma \Rightarrow \Delta}\ \text{Repl}_2^l$$

Theorem

The structural rules are admissible in $\mathbf{G3[mic]} + \mathit{Ref} + \mathit{Repl}_2^l$

Applications to logic with equality

$$\overline{\Gamma \Rightarrow \Delta, t = t} \quad \text{Ref}$$

$$\frac{r = s, \Gamma \Rightarrow \Delta, P[x/r]}{r = s, \Gamma \Rightarrow \Delta, P[x/s]} \ \text{Repl}_1^r \qquad \frac{s = r, \Gamma \Rightarrow \Delta, P[x/r]}{s = r, \Gamma \Rightarrow \Delta, P[x/s]} \ \text{Repl}_2^r$$

Applications to logic with equality

$$\overline{\Gamma \Rightarrow \Delta, t = t} \quad \overline{\text{Ref}}$$

$$\frac{r = s, \Gamma \Rightarrow \Delta, P[x/r]}{r = s, \Gamma \Rightarrow \Delta, P[x/s]} \; \text{Repl}_1^r \qquad \frac{s = r, \Gamma \Rightarrow \Delta, P[x/r]}{s = r, \Gamma \Rightarrow \Delta, P[x/s]} \; \text{Repl}_2^r$$

Theorem

The structural rules are admissible in $\mathbf{G3[mic]} + \overline{\mathit{Ref}} + \mathit{Repl}_1^r + \mathit{Repl}_2^r$

Applications to logic with equality

P not an equality, E an equality:

$$\frac{r = s, P[x/r], \Gamma \Rightarrow \Delta}{r = s, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Repl}_1^{l\neq} \qquad \frac{r = s, \Gamma \Rightarrow \Delta, E[x/r]}{r = s, \Gamma \Rightarrow \Delta, E[x/s]}\ \text{Repl}_1^{r=}$$

Applications to logic with equality

P not an equality, E an equality:

$$\frac{r = s, P[x/r], \Gamma \Rightarrow \Delta}{r = s, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Repl}_1^{l\neq} \qquad \frac{r = s, \Gamma \Rightarrow \Delta, E[x/r]}{r = s, \Gamma \Rightarrow \Delta, E[x/s]}\ \text{Repl}_1^{r=}$$

$$\frac{s = r, P[x/r], \Gamma \Rightarrow \Delta}{s = r, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Repl}_2^{l\neq} \qquad \frac{s = r, \Gamma \Rightarrow \Delta, E[x/r]}{s = r, \Gamma \Rightarrow \Delta, E[x/s]}\ \text{Repl}_2^{r=}$$

Theorem

The structural rules are admissible in
$\mathbf{G3[mic]} + \overline{Ref} + Repl_1^{l\neq} + Repl_2^{l\neq} Repl_1^{r=} + Repl_2^{r=}$

Applications to logic with equality

Repl_1^{I+} and Repl_2^{I+} same as Repl_1^{I} and Repl_2^{I} except that their instances concerning an equality E are strengthened into:

$$\frac{s = r, E[x/s], E[x/r], \Gamma \Rightarrow \Delta}{s = r, E[x/s], \Gamma \Rightarrow \Delta} \qquad \text{and} \qquad \frac{r = s,\ E[x/s], E[x/r], \Gamma \Rightarrow \Delta}{r = s,\ E[x/s], \Gamma \Rightarrow \Delta}$$

respectively.

Applications to logic with equality

Repl_1^{l+} and Repl_2^{l+} same as Repl_1^l and Repl_2^l except that their instances concerning an equality E are strengthened into:

$$\frac{s = r, E[x/s], E[x/r], \Gamma \Rightarrow \Delta}{s = r, E[x/s], \Gamma \Rightarrow \Delta} \qquad \text{and} \qquad \frac{r = s,\ E[x/s], E[x/r], \Gamma \Rightarrow \Delta}{r = s,\ E[x/s], \Gamma \Rightarrow \Delta}$$

respectively.

Theorem

The structural rules are admissible in

$$\mathbf{G3[mic]} + \overline{Ref} + Repl_2^{l+} + \ Repl_2^r$$

The same holds for $\mathbf{G3[mic]} + \overline{Ref} + Repl_1^{l+} + \ Repl_1^r$.

Applications to logic with equality

Repl_1^{l+} and Repl_2^{l+} same as Repl_1^{l} and Repl_2^{l} except that their instances concerning an equality E are strengthened into:

$$\frac{s = r, E[x/s], E[x/r], \Gamma \Rightarrow \Delta}{s = r, E[x/s], \Gamma \Rightarrow \Delta} \qquad \text{and} \qquad \frac{r = s,\ E[x/s], E[x/r], \Gamma \Rightarrow \Delta}{r = s,\ E[x/s], \Gamma \Rightarrow \Delta}$$

respectively.

Theorem

The structural rules are admissible in

$$\mathbf{G3[mic]} + \overline{Ref} + Repl_2^{l+} + \ Repl_2^{r}$$

The same holds for $\mathbf{G3[mic]} + \overline{Ref} + Repl_1^{l+} + \ Repl_1^{r}$.

Problem Can we do without strengthening Repl_1^{l} and Repl_2^{l} into Repl_1^{l+} and Repl_2^{l+} ?

Applications to logic with equality

Let $\prec$ be any antisymmetric relation on terms, i.e. if $r \prec s$, then $s \not\prec r$.

Problem

Are the structural rules admissible over $\overline{\Gamma \Rightarrow \Delta, t = t}$ and

$$\frac{r = s, P[x/r], \Gamma \Rightarrow \Delta}{r = s, P[x/s], \Gamma \Rightarrow \Delta}\ r \prec s \qquad \frac{r = s, \Gamma \Rightarrow \Delta, P[x/r]}{r = s, \Gamma \Rightarrow \Delta, P[x/s]}\ r \prec s$$

$$\frac{s = r, P[x/r], \Gamma \Rightarrow \Delta}{s = r, P[x/s], \Gamma \Rightarrow \Delta}\ s \not\prec r \qquad \frac{s = r, \Gamma \Rightarrow \Delta, P[x/r]}{s = r, \Gamma \Rightarrow \Delta, P[x/s]}\ s \not\prec r$$

?

Does it depend on $\prec$?

Applications to Semantic Tableau Systems

Tableau System:

Can extend a branch by adding:

- $T.t = t$ (Tableau Reflexivity Rule)
- $T.P[x/r]$, if it contains $T.s = r$ and $T.P[x/s]$ (Tableau Replacement Rule)

Applications to Semantic Tableau Systems

Alternate Tableau System

Can close a branch if it contains $F.t = t$ and can extend a branch that contains $T.r = s$ or $T.s = r$ by adding:

- $T.P[x/r]$ if it contains $T.P[x/s]$ where P is any atomic formula
- $F.E[x/r]$ if it contains $F.E[x/s]$ where E is an equality

(Alternate Tableau Replacement Rule)

Applications to Semantic Tableau Systems

A closed semantic tableau for $T.\Gamma, F.\Delta$ corresponds to a derivation of $\Gamma \Rightarrow \Delta$ in **GK3c** $+$ Ref $+$ Repl.

Applications to Semantic Tableau Systems

A closed semantic tableau for $T.\Gamma, F.\Delta$ corresponds to a derivation of $\Gamma \Rightarrow \Delta$ in **GK3c** $+$ Ref $+$ Repl.

GK3c $+$ Ref $+$ Repl is complete

Applications to Semantic Tableau Systems

A closed semantic tableau for $T.\Gamma, F.\Delta$ corresponds to a derivation of $\Gamma \Rightarrow \Delta$ in **GK3c** $+$ Ref $+$ Repl.

GK3c $+$ Ref $+$ Repl is complete

Every system equivalent to **GK3c** $+$ Ref $+$ Repl is complete

Applications to Semantic Tableau Systems

A closed semantic tableau for $T.\Gamma, F.\Delta$ corresponds to a derivation of $\Gamma \Rightarrow \Delta$ in **GK3c** $+$ Ref $+$ Repl.

GK3c $+$ Ref $+$ Repl is complete

Every system equivalent to **GK3c** $+$ Ref $+$ Repl is complete

G3c $+$ Ref $+$ Repl_2^l is complete

Applications to Semantic Tableau Systems

Theorem

The Tableau System remains sound and complete even if the strictness condition is imposed on the non γ logical rules, including the Tableau Replacement Rule.

Applications to Semantic Tableau Systems

Theorem

The Tableau System remains sound and complete even if the strictness condition is imposed on the non γ logical rules, including the Tableau Replacement Rule.

Theorem

The Alternate Tableau System is complete and remains complete also when the Alternate Tableau Replacement Rule is applied to $T.P$ only when P is not an equality and the full strictness condition is imposed.

Applications to Semantic Tableau Systems

Theorem

The Alternate Replacement Rules can be made directional as in the Tableau System, i.e. only left-right replacement is allowed, provided it is applied to formulae of the form $T.P$ as well as $F.P$. Furthermore also the strictness condition can be imposed, except when the Alternate Replacement Rule is applied to signed formulae of the form $T.E$, where E is an equality.

THANK YOU !