Classification of C-minimal groups with quasi-orders

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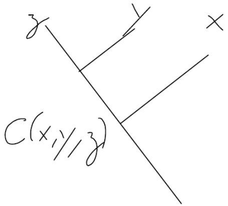
C-groups

A **C-relation** is a ternary relation which is interpretable in the set of branches of a meet-semilattice tree as follows: We say that $C(x, y, z)$ holds if and only if the meet of the branches x and y lies strictly below the meet of the branches y and z.



A **C-group** is a group with a C-relation which is compatible with the group operation i.e: $C(x, y, z) \Rightarrow C(vxu, vyu, vzu)$.

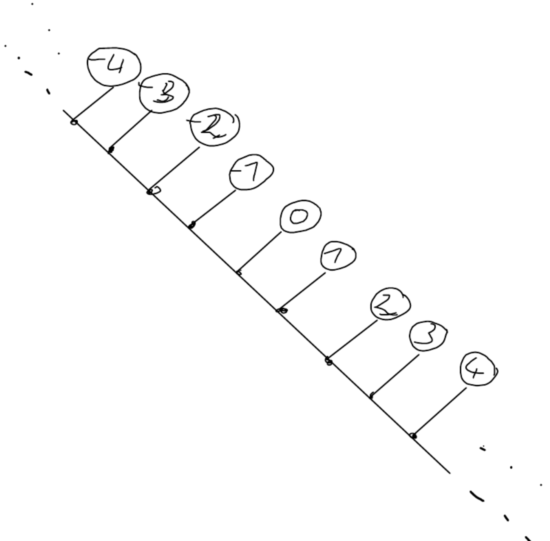
valued and ordered groups

- ordered group: $(G, \leq)$ such that $x \leq y \Rightarrow zx \leq zy \wedge xz \leq yz$.
- A valuation on a group G is a map $v : G \rightarrow \Gamma \cup \{\infty\}$ such that:
  - Γ is a totally ordered set, and this order is extended to $\Gamma \cup \{\infty\}$ by declaring $\gamma < \infty$ for all $\gamma \in \Gamma$.
  - For any $g \in G$, $v(g) = \infty \Leftrightarrow g = 1$ (the neutral element of the group).
  - For any $g, h \in G$, $v(gh) \geq min(v(g), v(h))$
  - For any $g \in G$, $v(g) = v(g^{-1})$

Example: the p-adic valuation on $\mathbb{Z}$, p a prime number: $v(a.p^k) = k$, where $gcd(a, p) = 1$.

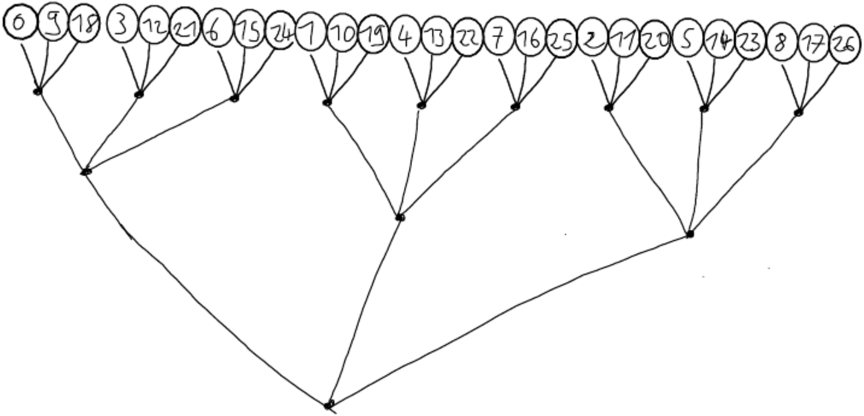
Fundamental C-groups

If $(G, \leq)$ is an ordered group, then $\leq$ induces a C-relation:
$C(x, y, z) : (y < x \wedge z < x) \vee (y = z \neq x)$.
Example: $(\mathbb{Z}, \leq)$:



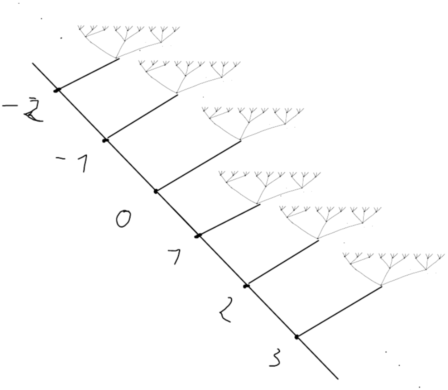
Fundamental C-groups

If (G, v) is a valued group, then v induces a C-relation:
$C(x, y, z) : v(y - z) > v(x - z)$.
Example: $(\mathbb{Z}/27\mathbb{Z})$ with the 3-adic valuation.



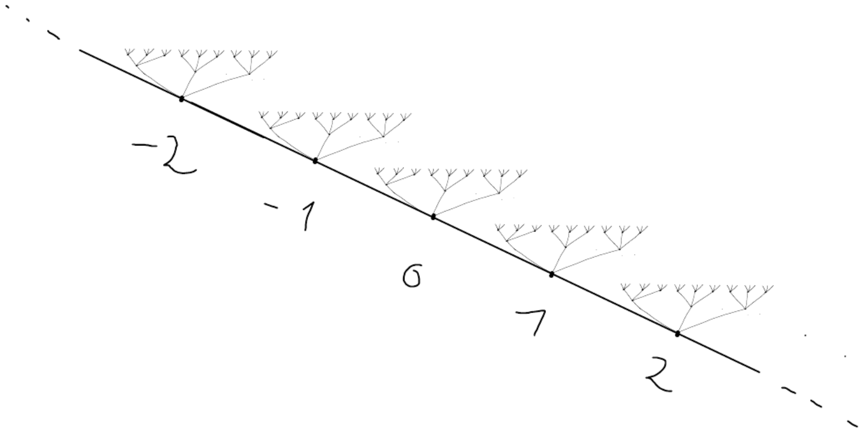
Products of C-groups

There is a natural notion of product of C-groups, which allows us to generate new examples of C-groups.



Example: $\mathbb{Z} \times (\mathbb{Z}/27\mathbb{Z})$.

welding



structure of a C-group

Theorem (Lehéricy 2016)

Any C-group can be obtained by lifting fundamental C-groups and then welding if necessary.

This means that valued and ordered groups are the building blocks of C-groups. Any C-group is a "mix" of ordered and valued groups. More precisely, if (G, C) is a C-group, you can partition G into "convex" parts, on each of which the C-relation either comes from a valuation or from an order.

C-minimality

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- Theorem (Lehéricy, 2018)

  Let (G, C) be a welding-free abelian C-minimal group. Then G is a finite product of o-minimal groups and C-minimal valued groups.

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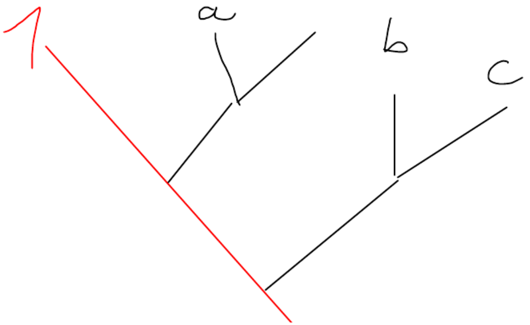
- Q.o. on G= an ordered partition of G.

Why quasi-orders?

If (G, C) is a C-group, then C induces a q.o. on G, called a **C-quasi-order**, by $x \precsim y \Leftrightarrow \neg C(x, y, 1)$. One compares two elements by looking at where they meet the branch representing 1:



$a \precsim b \sim c$

Why quasi-orders?

Q.o. are more intuitive than C-relations. This is because they are binary (and not ternary), and because they are similar to orders. Using the analogy between q.o.'s and orders, one can take methods from the well-known theory of ordered groups and use it for C-quasi-orders (e.g. notion of convexity, C-q.o. induced on a quotient...).

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