On the relations between the proof complexity measures of strongly equal k-tautologies in some proof systems.

Anahit Chubaryan and Garik Petrosyan
(speaker)

Department of Informatics and Applied Mathematics
Yerevan State University

Definitions

- Let E_k be the set $\left\{0, \frac{1}{\mathrm{k}-1}, \ldots, \frac{\mathrm{k}-2}{\mathrm{k}-1}, 1\right\}$

Definitions

- Let E_k be the set $\left\{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$
- We use the well-known notions of propositional formula, which defined as usual from propositional variables with values from, parentheses (,), and logical connectives $\neg, \wedge, \supset, \vee$ every of which can be defined by different mode

Definitions

- Let E_k be the set $\left\{0, \frac{1}{\mathrm{k}-1}, \ldots, \frac{\mathrm{k}-2}{\mathrm{k}-1}, 1\right\}$
- We use the well-known notions of propositional formula, which defined as usual from propositional variables with values from, parentheses (,), and logical connectives $\neg, \&, \supset, \vee$ every of which can be defined by different mode
- Additionally we use two modes of exponential function $p^{\boldsymbol{\sigma}}$ and introduce the additional notion of formula: for every formulas A and B the expression $\boldsymbol{A^B}$ (for both modes) is formula also

Definitions

- In the considered logics either only 1 or every of values $\frac{1}{2} \leq \frac{\boldsymbol{i}}{\mathbf{k-1}} \leq 1$ can be fixed as designated values

Definitions

- Definitions of main logical functions are:

$\boldsymbol{p} \vee \boldsymbol{q} = max(p, q)$ (1) disjunction or

$\boldsymbol{p} \vee \boldsymbol{q} = [(k-1)(p+q)](mod\ k)/(k-1)$ (2) disjunction

$\boldsymbol{p} \& q = min(p, q)$ (1) conjunction or

$\boldsymbol{p} \& q = \max(\boldsymbol{p} + q - 1, 0)$ (2) conjunction

Definitions

- For implication we have two following versions:

$$\boldsymbol{p} \supset \boldsymbol{q} = \begin{cases} 1, & for\ \ p \leq q \\ 1 - p + q, & for\ \ p > q \end{cases}$$

(1) Łukasiewicz's implication or

$$\boldsymbol{p} \supset \boldsymbol{q} = \begin{cases} 1, & for \quad p \leq q \\ q, & for \quad p > q \end{cases}$$

(2) Gödel's implication

Definitions

- And for negation two versions also:

$\neg\boldsymbol{p} = 1 - p$ (1) Łukasiewicz's negation or

$\neg\boldsymbol{p} = ((k-1)p + 1)(mod\ k)/(k-1)$ (2) cyclically permuting negation

Definitions

- For propositional variable ***p*** and $\boldsymbol{\delta}=\frac{i}{k-1}$ (0≤*i*≤*k*-1) additionally "exponent" functions are defined in

$\boldsymbol{p^{\delta}}$ as $(p \supset \delta) \& (\delta \supset p)$ with implication (1) exponent,

$\boldsymbol{p^{\delta}}$ as *p* with (k-1)–i negations. (2) exponent.

Note, that both (1) exponent and (2) exponent are not new logical functions.

Definitions

- If we fix "**1**" (every of values $\frac{1}{2} \leq \frac{i}{\mathrm{k}-1} \leq 1$) as designated value, than a formula ϕ with variables $p_1, p_2, \ldots p_n$ is called ***1*-*k*-tautology** (***≥1/2-k-tautology***) if for every $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_k^n$ assigning δ_j ($1 \leq j \leq n$) to each p_j gives the value 1 (or some value $\frac{1}{2} \leq \frac{i}{\mathrm{k}-1} \leq 1$) of ϕ.

Sometimes we call ***1*-*k*-tautology** or ***≥1/2-k-tautology*** simply ***k*-tautology.**

Definitions

- For every propositional variable p in k-valued logic $p^0, p^{1/_{k-1}}, \ldots, p^{k-2/_{k-1}}$ and p^1 in sense of both exponent modes are the literals. The conjunct K (term) can be represented simply as a set of literals (no conjunct contains a variable with different measures of exponents simultaneously), and DNF can be represented as a set of conjuncts.

Definitions

- We call ***replacement-rule*** each of the following trivial identities for a propositional formula φ

for both conjunction and (1) disjunction

$$\varphi\&0 = 0\&\varphi = 0, \quad \varphi \vee 0 = 0 \vee \varphi = \varphi, \quad \varphi\&1 = 1\&\varphi = \varphi, \quad \varphi \vee 1 = 1 \vee \varphi = 1,$$

for (2) disjunction

$$\left(\varphi \vee \frac{i}{k-1}\right) = \left(\frac{i}{k-1} \vee \varphi\right) = \overbrace{\neg\neg \ldots \neg}^{i}\varphi \qquad (0 \le i \le k-1),$$

Definitions

for (1) implication

$\varphi \supset 0 = \bar{\varphi}$ with (1) negation, $0 \supset \varphi = 1$, $\varphi \supset 1 = 1$, $1 \supset \varphi = \varphi$,

for (2) implication

$\varphi \supset 1 = 1$, $0 \supset \varphi = 1$, $\varphi \supset 0 = \overline{sg}\varphi$, where $\overline{sg}\varphi$ *is* 0 for φ>0 and 1 for φ=0,

Definitions

for (1) negation

¬(i/k-1)=1-i/k-1 (0≤i≤k-1), $\neg\boldsymbol{\psi} = \boldsymbol{\psi}$,

for (2) negation

¬(i/k-1)=i+1/k-1 (0≤i≤k-2), $\neg\mathbf{1} = \mathbf{0}$, $\overbrace{\neg\neg\ldots\neg}^{\boldsymbol{k}}\boldsymbol{\psi} = \boldsymbol{\psi}$.

Definitions

- Application of a replacement-rule to some word consists in replacing of its subwords, having the form of the left-hand side of one of the above identities, by the corresponding right-hand side

Definitions

- We call ***auxiliary relations for replacement*** each of the following trivial identities for a propositional formula φ

for both variants of conjunction

$$\left(\varphi \& \frac{i}{k-1}\right) = \left(\frac{i}{k-1} \& \varphi\right) \leq \frac{i}{k-1} \qquad (1 \leq i \leq k-2),$$

for (1) implication

$$\left(\varphi \supset \frac{i}{k-1}\right) \geq \frac{i}{k-1} \text{ and } \left(\frac{i}{k-1} \supset \varphi\right) \geq \frac{k-(i+1)}{k-1} \qquad (1 \leq i \leq k-2),$$

for (2) implication

$$\left(\varphi \supset \frac{i}{k-1}\right) \geq \frac{i}{k-1} \qquad (1 \leq i \leq k-2), \quad \left(\frac{i}{k-1} \supset \varphi\right) \geq \varphi \qquad (1 \leq i \leq k-1).$$

Definitions

- Let φ be a propositional formula of *k*-valued logic, $P = \{p_1, p_2, \dots, p_n\}$ be the set of all variables of φ and $P' = \{p_{i_1}, p_{i_2}, \dots, p_{i_m}\}$ $(1 \le m \le n)$ be some subset of P

Definitions

Definition 1: Given $\tilde{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_m) \in E_{\mathrm{k}}^{m}$, the conjunct $K^{\sigma} = \{{p_{i_1}}^{\sigma_1}, {p_{i_2}}^{\sigma_2}, \dots, {p_{i_m}}^{\sigma_m}\}$ is called $\varphi - \frac{i}{k-1}$-determinative $(0 \leq i \leq k-1)$, if assigning σ_j $(1 \leq j \leq m)$ to each $p_{i_j}$and successively using replacement-rules and, if it is necessary, the auxiliary relations for replacement also, we obtain the value $\frac{i}{k-1}$ of φ independently of the values of the remaining variables.

Every $\varphi - \frac{i}{k-1}$ $-$determinative conjunct is called also φ-determinative or determinative for φ.

Definitions

Definition 2. A DNF $D = \{K_1, K_2, \dots, K_j\}$ is called determinative DNF (dDNF) for φ if $\varphi = D$ and if "1" (every of values $\frac{1}{2} \leq \frac{i}{\mathrm{k-1}} \leq 1$) is (are) fixed as designated value, then every conjunct $K_i(1 \leq i \leq j)$ is 1-determinative ($\frac{i}{\mathrm{k-1}} -$ determinative from indicated intervale) for φ.

Definitions

Main Definition. *The k- tautologies φ and ψ are strongly equal in given version of many-valued logic if every φ -determinative conjunct is also ψ -determinative and vice versa.*

Definitions

- We compare the proof complexities measures of strongly equal ***k- tautologies*** in different systems of some versions of MVL.

Definitions

- We compare the proof complexities measures of strongly equal ***k- tautologies*** in different systems of some versions of MVL.

- One of considered system is the following universal elimination system **UE** for all versions of MVL.

Definitions

- The axioms of Elimination systems $\mathbf{UE}$ aren't fixed, but for every formula $\boldsymbol{k-valued\ \varphi}$ each conjunct from some DDNF of $\boldsymbol{\varphi}$ can be considered as an axiom.

Definitions

- The axioms of Elimination systems $\mathbf{UE}$ aren't fixed, but for every formula $\boldsymbol{k-valued\ \varphi}$ each conjunct from some DDNF of $\boldsymbol{\varphi}$ can be considered as an axiom.

- For *k*-valued logic the inference rule is *elimination rule* (ε-rule)

$$\frac{K_0 \cup \{p^0\},\ K_1 \cup \left\{p^{\frac{1}{k-1}}\right\},\ \dots\ ,\ K_{k-2} \cup \left\{p^{\frac{k-2}{k-1}}\right\},\ K_{k-1} \cup \{p^1\},}{K_0 \cup K_1 \cup \dots \cup K_{k-2} \cup K_{k-1}}$$

where mutual supplementary literals (variables with corresponding (1) or (2) exponents) are eliminated.

Definitions

- A finite sequence of conjuncts such that every conjunct in the sequence is one of the axioms of **UE** or is inferred from earlier conjuncts in the sequence by ε-rule is called a proof in **UE**.

Definitions

- A finite sequence of conjuncts such that every conjunct in the sequence is one of the axioms of **UE** or is inferred from earlier conjuncts in the sequence by ε-rule is called a proof in **UE**.

- A DNF $D = \{K_1, K_2, \dots, K_l\}$ is *k*-tautologi if by using ε-rule can be proved the empty conjunct ($\emptyset$) from the axioms $\{K_1, K_2, \dots, K_l\}$.

Definitions

In the theory of proof complexity four main characteristics of the proof are:

Definitions

In the theory of proof complexity two main characteristics of the proof are:

- $\boldsymbol{t-}\mathbf{complexity}$, defined as the number of proof steps (length)

Definitions

In the theory of proof complexity two main characteristics of the proof are:

- $\boldsymbol{t}-\mathbf{complexity}$ (length) , defined as the number of proof steps

- $\boldsymbol{l}-\mathbf{complexity}$ (size), defined as total number of proof symbols

Definitions

In the theory of proof complexity two main characteristics of the proof are:

- $\boldsymbol{s}-\mathbf{complexity}$ (space), informal defined as maximum of minimal number of symbols on blackboard, needed to verify all steps in the proof

Definitions

In the theory of proof complexity two main characteristics of the proof are:

- $\boldsymbol{s}-\mathbf{complexity}$ (space), informal defined as maximum of minimal number of symbols on blackboard, needed to verify all steps in the proof

- $\boldsymbol{w}-\mathbf{complexity}$ (width), defined as the maximum of widths of proof formulas.

Definitions

- Let $\boldsymbol{\Phi}$ be a proof system and $\boldsymbol{\varphi}$ be a *k*-tautology. We denote by $\boldsymbol{t_{\varphi}^{\Phi}}(\boldsymbol{l_{\varphi}^{\Phi}}, \boldsymbol{s_{\varphi}^{\Phi}}, \boldsymbol{w_{\varphi}^{\Phi}})$ the minimal possible value of $\boldsymbol{t - complexity}$ $(\boldsymbol{l - complexity}, \boldsymbol{s - complexity}, \boldsymbol{w - complexity})$ for all proofs of tautology $\boldsymbol{\varphi}$ in $\boldsymbol{\Phi}$.

Definitions

Theorem 1. The strongly equal *k*-tautologies have the same *t,l,s,w* complexities in the systems ***UE*** for all versions of MVL.

Definitions

- The situation for the systems ***L*** and ***G*** is the essentially other.

Definitions

- The situation for the systems ***L*** and ***G*** is the essentially other.

- For simplification of our result presentation, we demonstrate them only for 3-tautoogies.

Definitions

- ***For Łukasiewicz's 3-valued logic*** the following two 3-tautologies:

$A_n = (p^1 \,\&\, p^{1/2} \,\&\, p^0)^{1/2} \supset ((p^1 \,\&\, p^{1/2} \,\&\, p^0)^1 \supset (\overbrace{\neg\neg\ldots\neg}^{2n}(p^1 \vee p^{1/2} \vee p^0)))$ *with* (1) exponent, ($n \geq 0$),

$B_n = (p^1 \vee p^{1/2} \vee p^0) \,\&\, (\overbrace{\neg\neg\ldots\neg}^{2n}(p^1 \vee p^{1/2} \vee p^0))\,)))$ *with* (1) exponent, ($n \geq 0$),

Definitions

- ***For Gödel's 3-valued logic*** the following two 3-tautologies:

$C_n = \neg(\neg\neg p \& \neg p \& p) \supset ((\neg\neg p \& \neg p \& p) \supset (\overbrace{\neg\neg \ldots \neg}^{3n}(\neg\neg p \vee \neg p \vee p)))$ (n ≥0),

$D_n = (\neg\neg p \vee \neg p \vee p) \& (\overbrace{\neg\neg \ldots \neg}^{3n}(\neg\neg p \vee \neg p \vee p))$ *)))* (n ≥0).

Definitions

Theorem 2. a)

$$\boldsymbol{t}^{\boldsymbol{L}}_{\mathrm{A_n}} = O(1), \quad \boldsymbol{l}^{\boldsymbol{L}}_{\mathrm{A_n}} = O(n)$$

$$\boldsymbol{t}^{\boldsymbol{L}}_{\mathrm{B_n}} = \boldsymbol{\Omega}(\boldsymbol{n}), \quad \boldsymbol{l}^{\boldsymbol{L}}_{\mathrm{B_n}} = \Omega(n^2).$$

b)

$$\boldsymbol{t}^{\boldsymbol{G}}_{\mathrm{Cn}} = O(1), \quad \boldsymbol{l}^{\boldsymbol{G}}_{\mathrm{Cn}} = O(n),$$

$$\boldsymbol{t}^{\boldsymbol{G}}_{\mathrm{Dn}} = \boldsymbol{\Omega}(\boldsymbol{n}), \quad \boldsymbol{l}^{\boldsymbol{G}}_{\mathrm{Dn}} = \Omega(n^2).$$

Thank you for attention