First-Order Model Theory of Free Projective Planes

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Inspiration: Tarski's Problems

In this talk our setting will be first-order logic.

Conjecture (Tarski 1945)

Any two non-abelian free groups are elementary equivalent.

Conjecture (Tarski 1945)

The theory of a non-abelian free group is decidable.

Solution to Tarski's Problems

Sela (2006[1]) answered the first question positively, and Kharlampovich and Myasnikov (2006[2]) answered both questions independently, showing that the theory of free groups is decidable.

[1] Zlil Sela. *Diophantine Geometry over Groups. VI. The Elementary Theory of a Free Group*. Geom. Funct. Anal. **16** (2006), no. 3, 707-730.

[2] Olga Kharlampovich and Alexei Myasnikov. *Elementary Theory of Free Non-Abelian Groups*. J. Algebra **302** (2006), no. 2, 451-552.

Further Progress by Sela et al.

Sela (2013[3]) went deeper in the study of the model theory of free groups proving that this theory is (strictly) stable.

Various model theorists (most notably Perin, Pillay, Sklinos, and Tent) went on in this study proving many beautiful and deep results on the theory of free groups. Recently, a characterization of forking in free groups has been announced by Perin and Sklinos.

[3] Zlil Sela. *Diophantine Geometry over Groups VIII: Stability*. Ann. of Math. (2) **177** (2013), no. 3, 787-868.

Projective Planes

The topic of this talk is the model theory of free projective planes.

Definition

A partial plane is a system of points and lines satisfying:

(A) *there is at most one line through any two distinct points;*

(B) *there is at most one point common to any two distinct lines.*

We say that a partial plane is a projective plane if in (A)-(B) above we replace "at most" with "exactly one".

Free Extensions

Definition

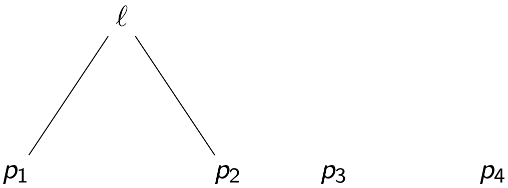
Given a partial plane P we can freely extend P to a projective plane $F(P)$ in the following way:

(1) *if you two points p_1 and p_2 are not joined by a line, then we add a new line $p_1 \vee p_2$ joining them;*

(2) *if you two lines ℓ_1 and ℓ_2 are parallel, then we add a new point $\ell_1 \wedge \ell_2$ passing through them.*

Repeat this ω many times and call the resulting plane $F(P)$.

In a Picture...



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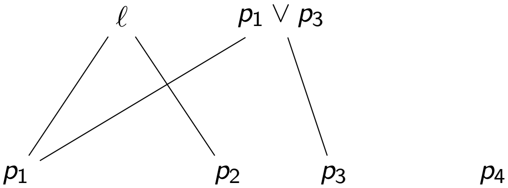

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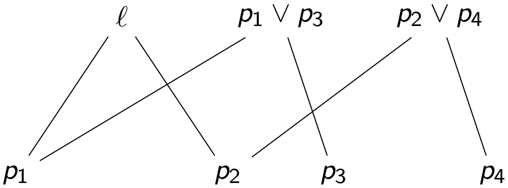

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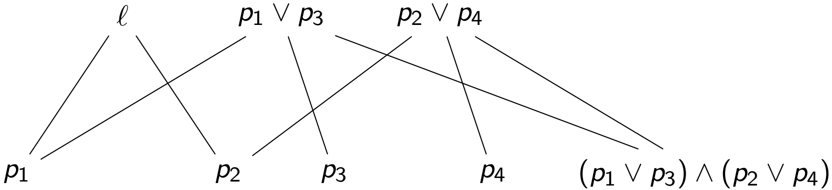

Free Projective Planes

Definition (M. Hall[4])

Given $4 \leqslant n \leqslant \omega$, we let π_0^n be the partial plane consisting of a line ℓ, $n-2$ points on ℓ and 2 points off of ℓ. We let $\pi^n = F(\pi_0^n)$ (the free extension of π^n), and call it the free projective plane of rank n.

Fact

For $4 \leqslant n < m \leqslant \omega$, $\pi^n \not\cong \pi^m$.

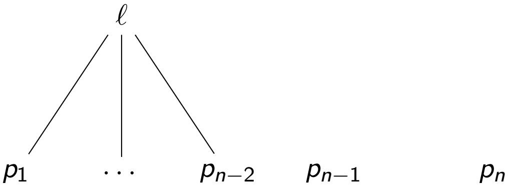

Figure: The partial plane π_0^n.

[4] Marshall Hall. *Projective Planes*. Trans. Amer. Math. Soc. **54** (1943), 229-277.

Open Projective Planes

Definition

Let P be a partial plane. We say that P is open if there is no finite subconfiguration A of P such that every element of A is incident with at least three elements of A.

Fact

Free projective planes are open.

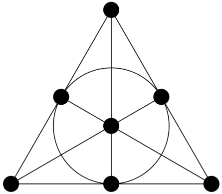

Figure: The Fano plane is not open.

HF-Constructions

Definition

Let $A \subseteq B$ be partial planes (in particular A can be $\emptyset$), we say that B is HF-constructible from (or over) A if there is a linear ordering $(B - A, <)$ such that for every $b \in B - A$ there are at most two elements of B such that they are incident with b and either from A or from $B - A$ and $<$-smaller than b.

Remark

Clearly π^n is HF-constructible from π_0^n.

Our Theorems I

Theorem (Hyttinen and P.)

The theory of open projective planes is complete.

Theorem (Hyttinen and P.)

If A and B are open projective planes and $A \subseteq B$, then A is elementary in B if and only if B is HF-constructible over A.

Our Theorems II

Theorem (Hyttinen and P.)

The theory of open projective planes is stable and not superstable.

Theorem (Hyttinen and P.)

For every infinite cardinality κ there are 2^κ non-isomorphic open projective planes of power κ.

Our Theorems III

Corollary (Hyttinen and P.)

The free projective planes $(\pi^n : 4 \leqslant n \leqslant \omega)$ *are all elementary equivalent, and they form an elementary chain with respect to the natural embeddings mapping* π_0^n *into* π_0^m*, for* $4 \leqslant n \leqslant m \leqslant \omega$*. Their common theory is the theory of open projective planes, and thus decidable, stable and not superstable.*

Theorem (Hyttinen and P.)

The free projective plane π^ω *is strongly type-homogeneous, i.e. for every tuple* a, b *in* π^ω *and finite set of parameter* A *in* π^ω*,* a *and* b *have the same type over* A *if and only if there is* $f \in Aut(\pi^\omega)$ *mapping* a *to* b *and fixing* A *pointwise.*

Our Theorems IV

We also characterized the forking independence relation in an arbitrary open projective plane (i.e., in an arbitrary model)!

Bibliography

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In preparation.