Modal Logics As Moodal Logics

Harold T. Hodes

Cornell University

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The Usual Approach

The usual approach to formalizing the modal logic K in a Natural Deduction format uses Necessitation and the K-rule.

$$\dfrac{\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}}{\Box\varphi}\,Nec \qquad\qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \Box(\psi\supset\varphi)\end{matrix}\quad\begin{matrix}\mathcal{D}_0\\ \Box\psi\end{matrix}}{\Box\varphi}\,K$$

$\mathcal{D}_0$ and $\mathcal{D}_1$ are any deductions with the indicated conclusions.
$\mathcal{D}$ is a deduction with no assumptions – Necessitation is a rule of proof rather than a rule of inference.

Are these proof-theoretically fundamental? No!

Moodal Markers, Marked Formulas

Our logical lexicon *LL*: $\{\bot, \top, \supset, \&, \vee, \Box, \Diamond\}$.
Fml $=$ the set of formulas generated by *LL* from some formula-constants.

$\mathbf{0}$ and $\mathbf{1}$ are our moodal markers.
A marked formula has the form $\mathbf{0}\varphi$ or $\mathbf{1}\varphi$ for $\varphi \in \mathit{Fml}$.
For $\Delta \subseteq \mathit{Fml}$ and $\mathbf{m} = \mathbf{0}$ or $\mathbf{1}$, $\mathbf{m}\Delta = \{\mathbf{m}\varphi \mid \varphi \in \Delta\}$.
MFml $=$ the set of m-formulas.

In $\mathbf{m}\varphi$, the marker $\mathbf{m}$ indicates a "mode of acceptance" of φ.
$\mathbf{0}$ marks acceptance as actually true;
$\mathbf{1}$ indicates acceptance as true at a (modally) accessible world.

Example of a deduction

Before defining $\Rightarrow_{IK}$, a sample IK-deduction, to motivate what will follow.

$\mathbf{0}\Box(\varphi \supset \psi), \mathbf{0}\Box\varphi \vdash \mathbf{0}\Box\psi$ will be witnessed by the following.

$$\dfrac{\dfrac{\dfrac{\mu_0{:}\mathbf{0}\Box(\varphi \supset \psi) \quad [\nu{:}\mathbf{1}\top]}{\mathbf{1}(\varphi \supset \psi)}\Box E \qquad \dfrac{\mu_1{:}\mathbf{0}\Box\varphi \quad [\nu{:}\mathbf{1}\top]}{\mathbf{1}\varphi}\Box E}{\mathbf{1}\psi}\mathbf{1}{\supset}E}{\mathbf{0}\Box\psi^{\ \nu}}\Box I$$

This deduction's dependency set = {[0,0,0],[0,1,0]}.

The associated model-theory might also motivate the definition of $\Rightarrow_{IK}$.

IK-Frames (from Plotkin & Sterling 1986)

$F = \langle W, R, \sqsubseteq \rangle$ is an IK-frame iff:

$R \subseteq W^2$, $\sqsubseteq\ \subseteq W^2$ is transitive and reflexive on W, and:

("right completeness") if $u \sqsubseteq u'$ and uRv then there is a v' so that $v \sqsubseteq v'$ and $u'Rv'$;

("left completeness") if $uRv \sqsubseteq v'$ then there is a u' so that $u \sqsubseteq u'Rv'$.

$$\begin{array}{ccc} u' - - - > \underline{v'} & \qquad & \underline{u'} - - - > v' \\ u \longrightarrow v & \qquad & u \longrightarrow v \\ \text{right} & \qquad & \text{left} \end{array}$$

u is a dead-end under F iff there is no v so that uRv.

$uR^{+}w$ iff for some v $u \sqsubseteq v$ and vRw.

IK Models, Formulas

$\mathcal{M} = \langle F, \mathcal{V} \rangle$ is a IK-model with signature S iff:

F is a IK-frame;

$\mathcal{V} : W \times S \to 2$, and for $u, v \in W$ and $\varphi \in S$, if $u \sqsubseteq v$ then $\mathcal{V}(u, \varphi) \leq \mathcal{V}(v, \varphi))$.

$\mathcal{M}, u \models (\varphi \supset \psi)$ iff: for any v, if $u \sqsubseteq v$ and $\mathcal{M}, v \models \varphi$ then $\mathcal{M}, v \models \psi$;

$\mathcal{M}, u \models \Box\varphi$ iff: for any u and w, if $uR^{+}w$ then $\mathcal{M}, w \models \varphi$;

$\mathcal{M}, u \models \Diamond\varphi$ iff: for some v, uRv and $\mathcal{M}, v \models \varphi$.

For $\Delta \subseteq Fml$ let $\mathcal{M}, u \models \Delta$ iff for every $\delta \in \Delta$ $\mathcal{M}, u \models \delta$.

Peristence, Marked Formulas

Persistence Lemma. If $\mathcal{M}, u \models \varphi$ and $u \sqsubseteq u'$ then $\mathcal{M}, u \models \varphi$.
Proof: If φ is $\Diamond\psi$, use the "right completeness" condition.

$\mathcal{M}, u \Vdash \mathbf{m}\varphi$ iff u is a dead-end, $\mathbf{m} = \mathbf{0}$, and $\mathcal{M}, u \models \varphi$.
$\mathcal{M}, u, v \Vdash \mathbf{m}\varphi$ iff uRv, and either $\mathbf{m} = \mathbf{0}$ and $\mathcal{M}, u \models \varphi$
or $\mathbf{m} = \mathbf{1}$, and $\mathcal{M}, v \models \varphi$.

$\mathcal{M}, u \Vdash \Gamma$ iff u is a dead-end and for every $\gamma \in \Gamma$ $\mathcal{M}, u \Vdash \gamma$.
$\mathcal{M}, u, v \Vdash \Gamma$ iff uRv, and for every $\gamma \in \Gamma$ $\mathcal{M}, u, v \Vdash \gamma$.

IK-valid Inferences

$\langle \Gamma, \chi \rangle$ is an inference iff $\Gamma \subseteq MFml$ and $\chi \in MFml$.

Consider an IK-model $\mathcal{M}$ and $u \in W^{\mathcal{M}}$.
$\langle \Gamma, \chi \rangle$ is $\mathcal{M}$-valid at u iff:
if $\mathcal{M}, u \Vdash \Gamma$ (so u is a dead-end under $F^{\mathcal{M}}$ and $\Gamma \subseteq \mathbf{0}Fml$),
then $\mathcal{M}, u \Vdash \chi$ so $\chi \in \mathbf{0}Fml$);
for every v, if $\mathcal{M}, u, v \models \Gamma$ (so u is not a dead-end under $F^{\mathcal{M}}$)
then $\mathcal{M}, u, v \models \chi$.

$\langle \Gamma, \chi \rangle$ is $\mathcal{M}$-valid iff it is is $\mathcal{M}$-valid at every $u \in W^{\mathcal{M}}$.

$\langle \Gamma, \chi \rangle$ is IK-valid iff for every IK-model $\mathcal{M}$, $\langle \Gamma, \chi \rangle$ is $\mathcal{M}$-valid.

Defining Being an IK-deduction

For $\nu \in Var$, $\chi \in MFml$,

$\nu{:}\chi$ is a tagged m-formula (a primitive type-assignment).

C is a context iff C is a single-valued set of tagged m-formulas.

A set of contexts is coherent iff its union is a context.

"$C \Rightarrow_{IK} \mathcal{D}{:}\chi$" means: relative to context C, $\mathcal{D}$ is an IK-deduction with conclusion χ.

Our definition will require that if $C \Rightarrow \mathcal{D}{:}\chi$ and $C' \Rightarrow \mathcal{D}{:}\chi$ then $C = C'$.

A deduction is a labeled tree, i.e. a function mapping a bare tree to labels of the latter's nodes.

One Conditional Introduction

We simultaneously define

$dpd(\mathcal{D})$ = the set of leaves of $\mathcal{D}$ on which $\mathcal{D}$ depends.

1 ⊃ Introduction If $C_0 \cup \{v{:}\mathbf{1}\varphi\} \Rightarrow \mathcal{D}_0{:}\mathbf{1}\psi$, $C_1 \Rightarrow \mathcal{D}_1{:}\mathbf{1}\theta$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow \mathcal{D}{:}\mathbf{1}(\varphi \supset \psi)$ for $\mathcal{D}$ as pictured.

$$\dfrac{\begin{array}{cc} [v{:}\mathbf{1}\varphi] & \\ \mathcal{D}_0 & \mathcal{D}_1 \\ \mathbf{1}\psi & \mathbf{1}\theta \end{array}}{\mathbf{1}(\varphi \supset \psi)}\ {}^{v}\ \mathbf{1}{\supset}I$$

$dpd(\mathcal{D}) = \{[0]\hat{\ }s \mid s \in dpd(\mathcal{D}_0),$
$\mathcal{D}_0(s) \neq v{:}\mathbf{1}\varphi\} \cup \{[1]\hat{\ }s \mid s \in dpd(\mathcal{D}_1)\}$.

Note the minor premise.

Transfer_0

The next rule is the only purely moodal rule of IK.

Transfer$_0$ If $\{\} \Rightarrow \mathcal{D}_0{:}\mathbf{0}\varphi$ and $C \Rightarrow \mathcal{D}_1{:}\mathbf{1}\theta$, then $C \Rightarrow \mathcal{D}{:}\mathbf{1}\varphi$ for $\mathcal{D}$ as pictured.

$$\dfrac{\begin{matrix}\mathcal{D}_0 & \qquad & \mathcal{D}_1 \\ \mathbf{0}\varphi & \qquad & \mathbf{1}\theta\end{matrix}}{\mathbf{1}\varphi}\,{\scriptstyle Trn_0}$$

The introduction rules for $\Box$ and $\Diamond$ will "freeze" the mood marked by **1** as iterable operators under the "home" mood marked by **0**; the elimination rules reverse this freezing.

Diamond Rules

$\Diamond$ Introduction If $C \Rightarrow \mathcal{D}_0{:}\mathbf{1}\varphi$, then $C \Rightarrow \mathcal{D}{:}\mathbf{0}\Diamond\varphi$ for this $\mathcal{D}$.

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ \mathbf{1}\varphi\end{matrix}}{\mathbf{0}\Diamond\varphi}\,\Diamond I$$

$\Diamond$ Elimination$_{\mathbf{0}}$ If $C_0, \nu{:}\mathbf{1}\varphi \Rightarrow \mathcal{D}_0{:}\mathbf{0}\psi$, $\mathcal{D}_0$ has a barrier with exception for $\nu{:}\mathbf{1}\varphi$, $C_1 \Rightarrow \mathcal{D}_1{:}\mathbf{0}\Diamond\varphi$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow \mathcal{D}{:}\mathbf{0}\psi$ for this $\mathcal{D}$.

$$\dfrac{\begin{matrix} \\ \mathcal{D}_1 \\ \mathbf{0}\Diamond\varphi\end{matrix} \qquad \begin{matrix}[\nu{:}\mathbf{1}\varphi] \\ \mathcal{D}_0 \\ \mathbf{0}\psi\end{matrix}}{\mathbf{0}\psi\ ^{\nu}}\,\Diamond E_{\mathbf{0}}$$

Barriers in Deductions

For $s \in dom(\mathcal{D})$, s is open in $\mathcal{D}$ iff for every leaf s' in $dom(\mathcal{D})$ such that $s \preceq s'$, $s' \in dpd(\mathcal{D})$.

A barrier in $\mathcal{D}$ with exception for $\nu{:}\mathbf{1}\varphi$ is an anti-chain $\{s_1, ..., s_m\} \subseteq dom(\mathcal{D})$ such that (i) for each $i \in (m)$ s_i is open in $\mathcal{D}$, the formula-label of s_i in $\mathcal{D}$ is in $\mathbf{0}Fml$, there is no $t \in dpd(\mathcal{D})$ with $s_i \preceq t$ and $\mathcal{D}(t) = \nu{:}\mathbf{1}\varphi$, but for some $t \in dpd(\mathcal{D})$ with $s_i \preceq t$ there is a ν' and φ' such that $\mathcal{D}(t) = \nu'{:}\mathbf{1}\varphi'$, and (ii) for every $t \in dpd(\mathcal{D})$ with $\mathcal{D}(t)$ of the form $\nu'{:}\mathbf{1}\varphi'$ and ν' is not ν, there is an $i \in (m)$ with $s_i \preceq t$.

Capturing the K-rule and Necessitation

$$\dfrac{\dfrac{\mu_0{:}\mathbf{0}\Box(\varphi \supset \psi) \quad [\nu{:}\mathbf{1}\top]}{\mathbf{1}(\varphi \supset \psi)}\Box E \qquad \dfrac{\mu_1{:}\mathbf{0}\Box\varphi \quad [\nu{:}\mathbf{1}\top]}{\mathbf{1}\varphi}\Box E}{\dfrac{\mathbf{1}\psi}{\mathbf{0}\Box\psi^{\ \nu}}\Box I}\mathbf{1}{\supset}E$$

$dpd = \{[0,0,0],[0,1,0]\}$; $barrier = \{\}$.

Assume that $\{\} \Rightarrow \mathcal{D}_0{:}\mathbf{0}\varphi$.

$$\dfrac{\dfrac{[\nu{:}\mathbf{1}\top] \qquad \begin{matrix}\mathcal{D}_0\\ \mathbf{0}\varphi\end{matrix}}{\mathbf{1}\varphi}Trn_0}{\mathbf{0}\Box\varphi^{\ \nu}}\Box I$$

Another Example; IK-consequence

Let $\Gamma \vdash_{IK} \chi$ iff for some C and $\mathcal{D}$, $C \Rightarrow \mathcal{D}{:}\chi$ and $ran(C) \subseteq \Gamma$.

$\vdash_{IK}$ is a finitary, monotonic structural (Tarskian) consequence relation on $MFml$.

$\vdash_{IK}$ is sound and complete relative to IK-models.

$\mathbf{0}\Diamond(\varphi \vee \psi) \vdash_{IK} \mathbf{0}(\Diamond\varphi \vee \Diamond\psi)$, witnessed by the following.

$$\dfrac{\mu{:}\mathbf{0}\Diamond(\varphi \vee \psi) \qquad \dfrac{[\nu_2{:}\mathbf{1}(\varphi \vee \psi)] \qquad \dfrac{\dfrac{[\nu_0{:}\mathbf{1}\varphi]}{\mathbf{0}\Diamond\varphi}\,\Diamond I}{\mathbf{0}(\Diamond\varphi \vee \Diamond\psi)}\,\mathbf{0}\vee I \qquad \dfrac{\dfrac{[\nu_1{:}\mathbf{1}\psi]}{\mathbf{0}\Diamond\psi}\,\Diamond I}{\mathbf{0}(\Diamond\varphi \vee \Diamond\psi)}\,\mathbf{0}\vee I}{\mathbf{0}(\Diamond\varphi \vee \Diamond\psi) \quad \nu_0,\nu_1}\,\mathbf{1}\vee E_{\mathbf{0}}}{\mathbf{0}(\Diamond\varphi \vee \Diamond\psi) \quad \nu_2}\,\Diamond E$$

Classical K

Define $\Rightarrow_{CK}$ by adopting the rules defining $\Rightarrow_{IK}$ and adding this rule.

0-Excluded Middle If $C_0, \nu_0{:}\mathbf{0}\varphi \Rightarrow_{CK} \mathcal{D}_0{:}\chi$, $C_1, \nu_1{:}\mathbf{0}\neg\varphi \Rightarrow_{CK} \mathcal{D}_1{:}\chi$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow_{CK} \mathcal{D}{:}\chi$, for $\mathcal{D}$ as pictured.

$$\dfrac{\begin{matrix}[\nu_0{:}\mathbf{0}\varphi] \\ \mathcal{D}_0 \\ \chi\end{matrix} \qquad \begin{matrix}[\nu_1{:}\mathbf{0}\neg\varphi] \\ \mathcal{D}_1 \\ \chi\end{matrix}}{\chi^{\ \nu_0,\nu_1}}\,\mathbf{0}EM$$

Let $dpd(\mathcal{D}) = (dpd(\mathcal{D}_0) - \{s \mid \mathcal{D}_0(s) = \nu_0{:}\mathbf{0}\varphi\}) \cup (dpd(\mathcal{D}_1) - \{s \mid \mathcal{D}_1(s) = \nu_0{:}\mathbf{0}\neg\varphi\})$.

Define $\vdash_{CK}$ from $\Rightarrow_{CK}$ in the obvious way.

The Systems IKD and IKD_box

Define $\Rightarrow_{IKD}$ and $\Rightarrow_{IKD_\Box}$ by adding the following rules, respectively.

$\mathbf{1}\top$ Introduction $\{\} \Rightarrow_{IKD} \mathcal{D}{:}\mathbf{1}\top$ for $\mathcal{D} = \{\langle [\,], \mathbf{1}\top \rangle\}$. A picture:

$$\overline{\mathbf{1}\top}^{\,IKD}$$

Strengthened $\mathbf{1}\bot E$ If $C, \nu{:}\mathbf{1}\top \Rightarrow_{IKD_\Box} \mathcal{D}_0{:}\mathbf{1}\bot$, $\mathcal{D}_0$ has a barrier with exception for $\nu{:}\mathbf{1}\top$, and $\theta \in \mathit{Fml}$, then $C \Rightarrow_{IKD_\Box} \mathcal{D}{:}\mathbf{0}\theta$ for $\mathcal{D}$ as pictured.

$$\dfrac{\begin{array}{c}[\nu{:}\mathbf{1}\top]\\ \mathcal{D}_0\\ \mathbf{1}\bot\end{array}}{\mathbf{0}\theta\ ^{\nu}}\,S\mathbf{1}\bot E$$

Let $dpd(\mathcal{D}) = \{[0]\hat{\ }s \mid s \in dpd(\mathcal{D}_0), \mathcal{D}_0(s) \neq \nu{:}\mathbf{1}\top\}$.

The Consequence Relations for KD and IKD_box

Define $\vdash_{IKD}$ $[\vdash_{IKD_\Box}]$ from $\Rightarrow_{IKD}$ $[\Rightarrow_{IKD_\Box}]$ in the obvious ways.

Observations (1) $\vdash_{IKD} \mathbf{0}\Diamond\top$.

(2) If $C \Rightarrow_{IKD_\Box} \mathcal{D}_0{:}\mathbf{0}\Box\bot$ and $\theta \in \mathit{Fml}$ then $C \Rightarrow_{IKD_\Box} \mathcal{D}{:}\mathbf{0}\theta$ for this $\mathcal{D}$.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \mathbf{0}\Box\bot\end{array} \qquad [\nu{:}\mathbf{1}\top]}{\mathbf{1}\bot}\,\Box E}{\mathbf{0}\theta}\,S\mathbf{1}\bot E\ \ \nu$$

(3) $\vdash_{IKD_\Box} \subsetneq \vdash_{IKD}$.

IKT_box, IKT_diamond

Define $\Rightarrow_{IKT_\Box}$ and $\Rightarrow_{IKT_\Diamond}$ by adding these rules, respectively.

0 Introduction If $C, \nu{:}\mathbf{1}\top \Rightarrow_{IKT_\Box} \mathcal{D}_0{:}\mathbf{1}\varphi$ and $\mathcal{D}_0$ has a barrier with exception for $\nu{:}\mathbf{1}\top$, then $C \Rightarrow_{IKT_\Box} \mathcal{D}{:}\mathbf{0}\varphi$ for this $\mathcal{D}$.

$$\dfrac{\begin{array}{c}[\nu{:}\mathbf{1}\top]\\ \mathcal{D}_0\\ \mathbf{1}\varphi\end{array}}{\mathbf{0}\varphi}\,{}^{\nu}\ \mathbf{0}I$$

0 Elimination If $C_1 \Rightarrow_{IKT_\Diamond} \mathcal{D}_1{:}\mathbf{0}\varphi$, $C_0, \nu{:}\mathbf{1}\varphi \Rightarrow_{IKT_\Diamond} \mathcal{D}_0{:}\mathbf{0}\theta$, $\mathcal{D}_0$ has a barrier with exception for $\nu{:}\mathbf{1}\varphi$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow_{IKT_\Diamond} \mathcal{D}{:}\mathbf{0}\theta$ for this $\mathcal{D}$.

$$\dfrac{\begin{array}{c}\\ \mathcal{D}_1\\ \mathbf{0}\varphi\end{array}\qquad\begin{array}{c}[\nu{:}\mathbf{1}\varphi]\\ \mathcal{D}_0\\ \mathbf{0}\theta\end{array}}{\mathbf{0}\theta}\,{}^{\nu}\ \mathbf{0}E$$

The Consequence-relations for them

Define $\vdash_{IKT_\Box}$ $[\vdash_{IKT_\Diamond}]$ from $\Rightarrow_{IKT_\Box}$ $[\Rightarrow_{IKT_\Diamond}]$ in the obvious ways.

Observations. (1) $\vdash_{IKD} \subsetneqq \vdash_{IKT_\Diamond}$. (2) $\vdash_{IKD_\Box} \subsetneq \vdash_{IKT_\Box}$. (3) $\vdash_{IKD_\Box} \subsetneq \vdash_{IKT_\Diamond}$.

ENOUGH for 15 minutes!