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Regarding Aristotelian Logic as a Sheffer Stroke Basic Algebra

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What we call today Aristotelian logic, it could be especially seen as the theory of the syllogisms.

The first systematic approach to the syllogisms dates back to the philosopher Aristotle who searched them in the scope of reasoning and inference as a logical system in the Prior Analytics [J. Barnes, 1984].

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At the end of 1800s, Lewis Caroll used diagrammatic methods to analyze the Aristotelian syllogisms in his book [L. Caroll, 1896]. In addition, Łukasiewicz interested with this topic comprehensively and he looked at this topic from the point of view of mathematical foundations in the middle of 1900s [J. Łukasiewicz, 1957]. These constitute the bases of modern mathematical works on categorical syllogisms.

Nowadays, the topic is studied extensively and investigated with different approaches.

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Nowadays, the topic is studied extensively and investigated with different approaches. For example, Stanley Burris examined syllogistic logic by using Boolean Algebras [S. Burris, 2013], Senturk and Oner examined by using Heyting Algebras [Senturk and Oner, 2016] and Esko Turunen used MV-Algebras for Peterson Intermediate Syllogisms [E. Turunen, 2014]. And also, syllogisms are used recently in different areas like as in computer science[I. Pratt-Hartmann and L. S. Moss, 2009], in artificial intelligence [B. Kumova and H. Cakir, 2010], in engineering [B. A. Kulik, 2001], in traffic control systems [J. Niittymäki and E. Turunen, 2003] etc.

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In this work, our first aim is to construct a bridge between Sheffer stroke basic algebra and categorical syllogisms together with a representation of syllogistic arguments by using sets in SLCD (Syllogistic Logic with Caroll Diagrams)[Senturk and Oner, 2018].

A categorical syllogistic system consists of 256 syllogistic moods, 15 of which are unconditionally and 9 are conditionally; in total 24 of them are valid. Those syllogisms in the conditional group are also said to be *strengthened*, or valid under *existential import*, which is an explicit assumption of existence of some S, M or P. So, we add a rule, which is "*Some* X *is* X when X exists", to SLCD. Therefore, we obtain the formal system SLCD† from SLCD.

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  This means that SLCD is sound and complete. And also,
- We define a Sheffer stroke algebra by using sets which is obtained from syllogistic arguments.

Preliminaries

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A categorical syllogism can be thought as a logical argument: It consists of two logical propositions called premises and a logical conclusion, where the premises and the conclusion have a quantified relationship between two objects which are given in *Table* 1. A syllogistic proposition or Aristotelian categorical proposition indicates a quantified relationship between two objects. There are four different types of propositions presented as follows:

Table 1: Aristotle's Syllogistic Propositions

Table: Aristotle's Syllogistic Propositions

Symbol	Statements	Generic Term
A	All X are Y	Universal Affirmative
E	No X are Y	Universal Negative
I	Some X are Y	Particular Affirmative
O	Some X are not Y	Particular Negative

We use S (for Subject term), M (for Middle term) and P (for Predicate term). That is, if there is a quantified relation between M and P (is said Major Premise), and a quantified relation between M and S (is said Minor Premise), then we deduce any result about a quantified relation between S and P (is said Conclusion).

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We assume that the relations between M and P, and between M and S hold. If we cannot contradict with certain relation between S and P does not hold, then the syllogism is valid. Otherwise, the syllogism is invalid.

Syllogisms are grouped into distinct four subgroups which are traditionally called *Figures* [E. Turunen, 2014]:

Figures

Figure I

A quantity Q_1 of M are P (Major Premise)

A quantity Q_2 of S are M (Minor Premise)

A quantity Q_3 of S are P (Conclusion)

Figures

Figure II

A quantity Q_1 of P are M (Major Premise)

A quantity Q_2 of S are M (Minor Premise)

A quantity Q_3 of S are P (Conclusion)

Figures

Figure III

A quantity Q_1 of M are P (Major Premise)

A quantity Q_2 of M are S (Minor Premise)

A quantity Q_3 of S are P (Conclusion)

Figures

Figure IV

A quantity Q_1 of P are M (Major Premise)

A quantity Q_2 of M are S (Minor Premise)

A quantity Q_3 of S are P (Conclusion)

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A syllogism is determined by using not only its mood but also its figure. And they are examined in terms of whether it is valid or not. So, we have some common properties which are called rules of deduction for getting valid syllogisims.

The rules of deduction of categorical syllogisms are the following:

Step 1: Relating to premises irrespective of conclusion or figure:

(a) No inference can be made from two particular premises.

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Step 2: Relating to propositions irrespective of figure:

(a) If one premise is particular, the conclusion must be particular.

(b) If one premise is negative, the conclusion must be negative.

Step 3: Relating to distribution of terms:

(a) The middle term must be distributed at least once.

(b) A predicate distributed in the conclusion must be distributed in the major premise.

(c) A subject distributed in the conclusion must be distributed in the minor premise.

We use $\vdash$ symbol for valid syllogisms. For example, the syllogism

$$\boldsymbol{A}_{MP}, \boldsymbol{A}_{SM} \vdash \boldsymbol{A}_{SP}$$

consists of from left to right *major premise*, *minor premise* and *conclusion*, respectively. Its mood is ***AAA***, and it has first figure.

A categorical syllogistic system has 256 moods for all figures. 15 of them are unconditionally and 9 of them are conditionally, totally 24 of them are valid forms. We have *unconditional valid forms of syllogism* in Table 2. It means that these forms are valid without any condition in Syllogism.

Table 2: Unconditionally Valid Forms

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Figure I	Figure II	Figure III	Figure IV
$\boldsymbol{A}_{MP}, \boldsymbol{A}_{SM} \vdash \boldsymbol{A}_{SP}$	$\boldsymbol{E}_{PM}, \boldsymbol{A}_{SM} \vdash \boldsymbol{E}_{SP}$	$\boldsymbol{I}_{MP}, \boldsymbol{A}_{MS} \vdash \boldsymbol{I}_{SP}$	$\boldsymbol{A}_{PM}, \boldsymbol{E}_{MS} \vdash \boldsymbol{E}_{SP}$
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Syllogistic forms in *Table* 3 are *valid syllogistic forms depending on some conditions*. If these conditions hold, then these syllogistic forms are valid.

Table 3: Conditionally Valid Forms

Table: Conditionally Valid Forms

Figure I	Figure II	Figure III	Figure IV	Necessary Condition
$\boldsymbol{A}_{MP}, \boldsymbol{A}_{SM} \vdash \boldsymbol{I}_{SP}$	$\boldsymbol{A}_{PM}, \boldsymbol{E}_{SM} \vdash \boldsymbol{O}_{SP}$		$\boldsymbol{A}_{PM}, \boldsymbol{E}_{MS} \vdash \boldsymbol{O}_{SP}$	S exists
$\boldsymbol{E}_{MP}, \boldsymbol{A}_{SM} \vdash \boldsymbol{O}_{SP}$	$\boldsymbol{E}_{PM}, \boldsymbol{A}_{SM} \vdash \boldsymbol{O}_{SP}$			S exists
		$\boldsymbol{A}_{MP}, \boldsymbol{A}_{MS} \vdash \boldsymbol{I}_{SP}$	$\boldsymbol{E}_{PM}, \boldsymbol{A}_{MS} \vdash \boldsymbol{O}_{SP}$	M exists
		$\boldsymbol{E}_{MP}, \boldsymbol{A}_{MS} \vdash \boldsymbol{O}_{SP}$		M exists
			$\boldsymbol{A}_{PM}, \boldsymbol{A}_{MS} \vdash \boldsymbol{I}_{SP}$	P exists

Remark

The syllogisms in the Table 2 are referred to simply as **syllogisms**, those in Table 3 are referred as **strengthened syllogisms**.

Mnemonic Names of All Valid Forms

Here are the traditional mnemonic names of 24 of the forms, arranged by figures:

1	2	3	4
Barbara	Cesare	Darapti $*$	Bramantip $*$
Celarent	Camestres	Felapton $*$	Camenes
Darii	Festino	Disamis	Dimaris
Ferio	Baroco	Datisi	Fesapo $*$
Barbari †	Camestrop †	Bocardo	Fresison
Celaront †	Cesaro †	Ferison	Camenop †

Carroll's Diagrams and The Elimination Method

Carroll's diagrams, thought up in 1884, are Venn-type diagrams where the universes are represented by a square [L. Caroll, 1896]. Nevertheless, it is not clear whether Carroll studied his diagrams independently or as a modification of John Venn's. Still, Carroll's scheme looks like a sophisticated method summing up several developments that have been introduced by researchers stuying in this area.

Let X and Y be two terms and let X' and Y' be the complements of X and Y, respectively. For two-terms, Carroll divides the square into four cells, and he gets the so-called bilateral diagram, as shown in below:

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Y'	$X'Y'$	XY'
Y	$X'Y$	XY

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Each of these four cells can have three possibilities, when we explain the relations between two terms. They can be 0 or 1 or *blank*. In this method, 0 means that there is no element intersection cell of two elements, 1 means that it is not empty and *blank* cell means that we don't have any information about the content of the cell, therefore it could be 0 or 1.

As above method, let X, Y, and M be three terms and X', Y', and M' be their respective complements. To examen all relations between three terms, he added one more square in the middle of bilateral diagram which is called the trilateral diagram, as the following:

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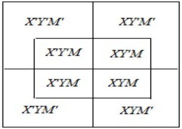

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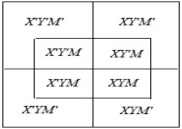

Each cell in a trilateral diagram is marked with a 0, if there is no element and is marked with a **I** if it is not empty and another using of **I**, it could be on the line where the two cell is intersection, this means that at least one of these cells is not empty. So, **I** is different from 1. In addition to these,if any cell is **blank**, it has two possibilities, 0 or **I**.

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Second Rule: If the quarter of trilateral diagram contains a "**I**" in either cell, then it is certainly occuppied, and one may mark the corresponding quarter of the bilateral diagram with a "1" to indicate that it is occupied.

Third Rule: If the quarter of trilateral diagram contains two "0"s, one in each cell, then it is certainly empty, and one may mark the corresponding quarter of the bilateral diagram with a "0" to indicate that it is empty.

The Calculus System *SLCD* and Its Completeness

In this section, we correspond a set to each possible form of any syllogistic bilateral diagrams and also define universes of major and minor premises and conclusions in the categorical syllogisms. Moreover, we give a definition of a map which obtains a conclusion from two possible forms of premises. Then, we generalize it for conclusion of any two premises and also valid forms in syllogisms.

Our aim is to construct a complete bridge between *Sets* and *Aristotelian Logic*:

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Table: *The Paradigm for the Representation of Syllogistic Arguments by using Sets*

	LOGIC	DIAGRAMS	SETS
PREMISES	Propositions	$\xrightarrow{\textit{Translate}}$	Sets
			$\downarrow$
CONCLUSION	Propositions	$\xleftarrow{\textit{Translate}}$	Sets

Let X and Y be two terms and their complements are denoted by X' and Y', respectively. Assume that p_i shows a possible form of any bilateral diagram, such that $1 \leq i \leq k$, where k is the number of possible forms of bilateral diagram, as follows:

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Table: Bilateral diagram for a quantity relation between X and Y

p_i	X'	X
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Y	n_3	n_4

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where $n_1, n_2, n_3, n_4 \in \{0, 1\}$. Given throughout this paper the symbols $R_{(A)}$, $R_{(E)}$, $R_{(I)}$ and $R_{(O)}$ represent "*All*", "*No*", "*Some*" and "*Some* − *not*" statements, respectively.

"All S are P"

We examine *All S are P*, it means that there is no element in the intersection of S and P' cell. This is shown in the following bilateral diagram:

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Table: *Bilateral diagram for "All S are P"*

$R_{(A)} =$

	P'	P
S'		
S	0	

From the above Table, we obtain all possible bilateral diagrams having 0 in the intersection of S and P' cell:

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Table: *Possible forms of "All S are P"*

p_1	P'	P
S'	0	0
S	0	0

p_2	P'	P
S'	0	0
S	0	1

p_3	P'	P
S'	0	1
S	0	0

p_4	P'	P
S'	1	0
S	0	0

p_5	P'	P
S'	0	1
S	0	1

p_6	P'	P
S'	1	0
S	0	1

p_7	P'	P
s'	1	1
S	0	0

p_8	P'	P
S'	1	1
S	0	1

These tables show all possible forms of *"All S are P"*.

Now in order to correspond bilateral diagrams and sets, let us form a set consisting of numbers which correspond to possible forms that each bilateral diagram possesses. To do this, first we define the value which corresponds to the bilateral diagram.

Definition [A. E. Kulinkovich, 1979]

Let r_j^{val} denote the value corresponding to a possible bilateral diagram p_j and n_i is the value that the i-th cell possesses, then the value of this possible bilateral diagram is calculated by using the formula

$$r_j^{val} = \sum_{i=1}^{4} 2^{(4-i)} n_i, \quad 1 \leq j \leq k,$$

where k is the number of all possible forms.

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Definition

Let R^{set} be the set of the values which correspond to all possible forms of any bilateral diagram; that is
$R^{set} = \{r_j^{val} : 1 \leq j \leq k, k \text{ is the number of all possible forms}\}$.
The set of all these R^{set}'s is denoted by $\mathcal{R}^{Set}$.

The set representations of all categorical propositions as follows:

The set representations of all categorical propositions as follows:

- *All X are Y:* It means that the intersection of X and Y' is empty set:

$R_{(A)} =$

	X'	X
Y'		0
Y		

Then the set representation of "*All X are Y*" is

$$R^{set}_{(A)} = \{0, 1, 2, 3, 8, 9, 10, 11\}.$$

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Y		

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$$R^{set}_{(A)} = \{0, 1, 2, 3, 8, 9, 10, 11\}.$$

- *No X are Y:* No element in the intersection cell of X and Y

$R_{(E)} =$

	X'	X
Y'		
Y		0

$$R^{set}_{(E)} = \{0, 2, 4, 6, 8, 10, 12, 14\}.$$

- *Some X are Y:* There is at least one element which belongs X and Y

$R_{(I)} =$

	X'	X
Y'		
Y		1

$$R_{(I)}^{set} = \{1, 3, 5, 7, 9, 11, 13, 15\}.$$

- *Some X are Y:* There is at least one element which belongs X and Y

$R_{(I)} =$

	X'	X
Y'		
Y		1

$$R^{set}_{(I)} = \{1, 3, 5, 7, 9, 11, 13, 15\}.$$

- *Some X are not Y:* If some elements of X are not Y, then they have to be in Y'.

$R_{(O)} =$

	X'	X
Y'		1
Y		

$$R^{set}_{(O)} = \{4, 5, 6, 7, 12, 13, 14, 15\}.$$

Example-Validity of AAA

If *All S are M* and *All M are P*, then *All S are P*. This syllogism, called *Barbara*, is valid. We show this truth by using elimination method from trilateral daigram to bilateral diagram.

All S are M: it means that the intersection of cell S and M' is 0 without any condition. It is shown as follows:

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	S'	S
M'		0
M		

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M'		0
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	P'	P
M'		
M	0	

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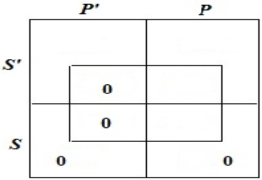

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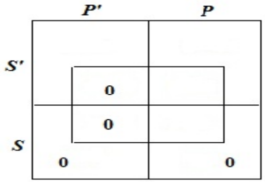

By the elimination method, we obtain the relation between S and P on the bilateral diagram:

$R_{(A)} =$

	P'	P
S'		
S	0	

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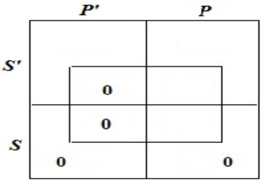

By the elimination method, we obtain the relation between S and P on the bilateral diagram:

$R_{(A)} =$

	P'	P
S'		
S	0	

This means *"All S are P"*. So, we can say that *this syllogism is valid*.

Example

Let p_i be one possible form of the bilateral diagram having a relation between P and M, and p_j be one possible form of the bilateral diagram having a relation between S and M. Then, we can obtain a relation between S and P. We take possible forms given as below:

$p_i =$

	P'	P
M'	0	0
M	1	1

and

$p_j =$

	S'	S
M'	0	0
M	1	0

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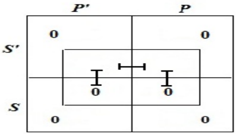

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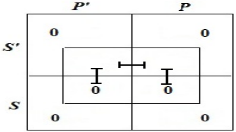

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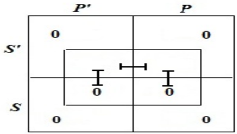

By using the elimination method, we can obtain a relation between S and P as below:

$p_I =$

	P'	P
S'	1	1
S	0	0

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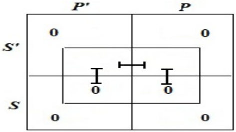

By using the elimination method, we can obtain a relation between S and P as below:

$p_l =$

	P'	P
S'	1	1
S	0	0

$r_i^{val} = 2$ corresponds to possible form p_i, and $r_j^{val} = 3$ corresponds to possible form p_j, we obtain that $r_l^{val} = 12$ corresponds to p_l that is a possible conclusion.

Question

Can we generalize it for all possible bilateral diagrams?

After these examples, we try to generalize them by formula. So, we define an operation and a theorem as follows:

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Definition

The syllogistic possible conclusion mapping, denoted $*$, is a mapping which gives us the deduction set of possible forms of major and minor premises sets.

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Definition

The syllogistic possible conclusion mapping, denoted $*$, is a mapping which gives us the deduction set of possible forms of major and minor premises sets.

Theorem

Let r_i^{val} and r_j^{val} correspond to the numbers of possible forms of major and minor premises, respectively. Then, $r_i^{val} * r_j^{val}$ equals the value given by the intersection of row and column numbers corresponding to r_i^{val} and r_j^{val} in Table 4.

We deal with the operation table below given by Kulinkovich [A. E. Kulinkovich, 1979]. It is used for finding valid syllogisms by means of set theoretical representation of bilateral diagrams.

Table 4: Operation Table

$*$	0	1	2	3	4	8	12	5	10	6	9	7	11	13	14	15
0	0															
1		1	4	5												
2		2	8	10												
3		3	12	H												
4					1	4	5									
8					2	8	10									
12					3	12	H									
5								1	4	5	5	5	5	5	5	5
10								2	8	10	10	10	10	10	10	10
6								3	12	9	6	11	14	7	13	15
9								3	12	6	9	7	13	11	14	15
7								3	12	13	7	H_4	H_3'	7	13	H_1'
11								3	12	14	11	H_3	H_4'	11	14	H_2'
13								3	12	7	13	7	13	H_4	H_3'	H_1'
14								3	12	11	14	11	14	H_3	H_4'	H_2'
15								3	12	15	15	H_1	H_2	H_1	H_2	H

In the above Table, considering possible conclusion mapping. But some possible forms of premises have more than one possible conclusions, given as below:

$$H = \{6,7,9,11,13,14,15\},\ H_1 = \{7,11,15\},\ H_1' = \{6,7,9,11,13,15\},$$
$$H_2 = \{13,14,15\},\ H_2' = \{11,14,15\},\ H_3 = \{6,7,11,14,15\},$$
$$H_3' = \{6,7,13,14,15\},\ H_4 = \{7,9,11,13,15\},\ H_4' = \{9,11,13,14,15\}$$

Indeed, the possible conclusion is an image of possible premises under a mapping.

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Definition

Universes of values sets of major premises, minor primises, and conclusions are denoted by $\mathcal{R}^{set}_{Maj}$, $\mathcal{R}^{set}_{Min}$ and $\mathcal{R}^{set}_{Con}$, respectively.

Let $R^{set}_{(k)}$ be an element of $\mathcal{R}^{set}_{Maj}$ and $R^{set}_{(l)}$ be an element of $\mathcal{R}^{set}_{Min}$. The main problem is what the conclusion of these premises is. In syllogistic, we have some patterns which are mentioned in Table 2 and Table 3 above. Now, we explain them by using bilateral diagrams with an algebraic approach.

Definition

The syllogistic mapping, denoted by $\circledast$, is a mapping which gives us the conclusion of the major and the minor premises as below:

	P'	P
M'		
M		

$\circledast$

	S'	S
M'		
M		

$=$

	P'	P
S'		
S		

Theorem

Let $R^{set}_{(k)} = \{r^{val}_{k_1}, \ldots, r^{set}_{k_n}\}$ and $R^{set}_{(l)} = \{r^{val}_{l_1}, \ldots, r^{val}_{l_t}\}$ two sets corresponding to the Major and the Minor premises. Then $\circledast : \mathcal{R}^{set}_{Maj} \times \mathcal{R}^{set}_{Min} \to \mathcal{R}^{set}_{Con}$

$$R^{set}_{(k)} \circledast R^{set}_{(l)} := \bigcup_{j=1}^{n} \bigcup_{i=1}^{t} r^{val}_{k_j} * r^{val}_{l_i}$$

is the conclusion of the premises $R^{set}_{(k)}$ and $R^{set}_{(l)}$.

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Let $R^{set}_{(k)} = \{r^{val}_{k_1}, \ldots, r^{set}_{k_n}\}$ and $R^{set}_{(l)} = \{r^{val}_{l_1}, \ldots, r^{val}_{l_t}\}$ two sets corresponding to the Major and the Minor premises. Then $\circledast : \mathcal{R}^{set}_{Maj} \times \mathcal{R}^{set}_{Min} \to \mathcal{R}^{set}_{Con}$

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is the conclusion of the premises $R^{set}_{(k)}$ and $R^{set}_{(l)}$.

Theorem

A syllogism is valid if and only if it is provable in *SLCD*.

For conditional valid forms, we need an addition rule which is *"Some X are X"*. We can use above Theorem by taking into consideration this rule.

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Definition

Let $SLCD^{\dagger}$ be the formal system which is obtained from $SLCD$ by addition of the rule:

$$\vdash \boldsymbol{I}_{XX}$$

to show the validity of strengthened formulas.

Definition

Let $R_{(k)}$ be the bilateral diagram presentation of the premise. The *transposition* of a premise is the symmetric positions with respect to the main diagonal. It is shown by $Trans(R_{(k)})$.

$$\begin{aligned} Trans : \mathcal{R}^{set} &\rightarrow \mathcal{R}^{set}, \\ R^{set}_{(k)} &\rightarrow Trans(R^{set}_{(k)}) = \{r^{val}_{k^T_1}, \ldots, r^{set}_{k^T_n}\}. \end{aligned}$$

Theorem

Let $R^{set}_{(k)} = \{r^{val}_{k_1}, \ldots, r^{set}_{k_n}\}$ and $R^{set}_{(l)} = \{r^{val}_{l_1}, \ldots, r^{val}_{l_t}\}$ be two sets to correspond to the Major and the Minor premises values sets and $R^{set}_{(s)} = \{r^{val}_{s_1}, \ldots, r^{set}_{s_m}\}$ be set to correspond to the constant set values which means "Some S are S", "Some M are M" and "Some P are P". Then $\circledast^{\dagger} : \mathcal{R}^{set}_{Maj} \times \mathcal{R}^{set}_{Min} \to \mathcal{R}^{set}_{Con}$

$$R^{set}_{(k)} \circledast^{\dagger} R^{set}_{(l)} := \begin{cases} \bigcup_{j=1}^{n} \bigcup_{i=1}^{t} \bigcup_{h=1}^{m} (r^{val}_{k_j} * (r^{var}_{s_h} * r^{var}_{l_i^T})), & \textit{If S exists,} \\ \bigcup_{j=1}^{n} \bigcup_{i=1}^{t} \bigcup_{h=1}^{m} (r^{val}_{k_j} * (r^{var}_{l_i} * r^{var}_{s_h})), & \textit{If M exists,} \\ \bigcup_{j=1}^{n} \bigcup_{i=1}^{t} \bigcup_{h=1}^{m} ((r^{var}_{s_h} * r^{val}_{k_j^T}) * r^{var}_{l_i}), & \textit{If P exists.} \end{cases}$$

is the conclusion of the premises $R^{set}_{(k)}$ and $R^{set}_{(l)}$ under the conditions *S exists*, *M exists* or *P exists*.

Theorem

A strengthened syllogism is valid if and only if it is provable in $SLCD^\dagger$.

In this section, we construct a Sheffer stroke basic algebra on categorical syllogisms by means of set theoretical representation of their bilateral diagrams. At first, we define $\wedge$ (meet) and $\vee$ (join) operators on the set of numbers corresponding to possible form of bilateral diagrams.

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Definition

Let $R^{(k)}$ and $R^{(l)}$ be elements of $\mathcal{R}$. Then the definitions of binary join and meet operations are as follows:

$$R_{(k)} \vee R_{(l)} := R^{set}_{(k)} \cup R^{set}_{(l)}$$

$$R_{(k)} \wedge R_{(l)} := R^{set}_{(k)} \cap R^{set}_{(l)}$$

Question

Which type algebraic structures are constructed by means of set theoretical representation of Aristotelian Logic?

Theorem

$< \mathcal{R}, \vee, \wedge >$ is a distributive lattice.

Corollary

$< \mathcal{R}^{set}_{Maj}, \cup, \cap >$, $< \mathcal{R}^{set}_{Min}, \cup, \cap >$ and $< \mathcal{R}^{set}_{Con}, \cup, \cap >$ are distributive lattices.

Now, we define an order relation on $\mathcal{R}^{set}$ as follows:

$$R^{set}_{(k)} \preceq R^{set}_{(l)} :\Leftrightarrow R^{set}_{(k)} \subseteq R^{set}_{(k)}.$$

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Theorem

$\mathcal{R}^{set}$ is partially ordered by the binary relation $\preceq$.

Let $(\mathcal{R}^{set}, \preceq)$ be a poset. The greatest element of $\mathcal{R}^{set}$ is $\{0, 1, \ldots, 15\}$, denoted by $\mathbf{1}$ and the least element is $\emptyset$, denoted by $\mathbf{0}$. We notice again that $\mathbf{0}$ and 0 are different from each other. Let R_k be any element of $\mathcal{R}$. Then we have

$$R_{(k)} \wedge \mathbf{0} = R^{set}_{(k)} \cap \emptyset = \emptyset = \mathbf{0}$$

and

$$R_{(k)} \vee \mathbf{1} = R^{set}_{(k)} \cup \{0, 1, \ldots, 15\} = \{0, 1, \ldots, 15\} = \mathbf{1}.$$

Definition

The complement of R, denoted by R^c, $R^c = \{0, 1, 2, \ldots, 15\} \setminus R$.

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$< \mathcal{R}, \vee, \wedge, ^c, \mathbf{0}, \mathbf{1} >$ is an *ortholattice*.

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$< \mathcal{R}, \vee, \wedge, ^c, \mathbf{0}, \mathbf{1} >$ is an *orthomodular lattice*.

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Question

Why do we construct a Sheffer stroke basic algebra on set theoretical representation of Aristotelian Logic?

Our main idea

The Sheffer stroke operation has a crucial role for computer systems. To put it more explicitly, it has an useful application in chip technology as it allows to have all diods on the chip forming processor in a computer in a uniform manner. Hence, this is cheapher and simpler than to use different diods for other logical connectives such as conjunction, disjunction, negation and etc.

Oner and Senturk defined an alternative signature $\{|\}$ which composes of only the Sheffer stroke operation [Oner T. and Senturk I., 2017]. Therefore, it is of some importance to analyze the Sheffer stroke reduct in a general setting for basic algebras. Now, we give the following axiomatic system for Sheffer stroke reduction of basic algebras.

Definition [Oner T. and Senturk I., 2017]

An algebra $(A, |)$ of type $< 2 >$ is called a *Sheffer stroke basic algebra* if it satisfies the following identities:
$(SH1)$ $(a|(a|a))|(a|a) = a$,
$(SH2)$ $(a|(b|b))|(b|b) = (b|(a|a))|(a|a)$,
$(SH3)$ $(((a|(b|b))|(b|b))|(c|c))|((a|(c|c))|(a|(c|c))) = a|(a|a)$.

Definition

Let $R_{(k)}$ and $R_{(l)}$ be elements of $\mathcal{R}$. Then the definition of Sheffer stroke operation is as follow:

$$R_{(k)}|R_{(l)} := (\mathbf{1} \setminus R^{set}_{(k)}) \cup (\mathbf{1} \setminus R^{set}_{(l)})$$

Theorem

$< \mathcal{R}, |, \mathbf{0}, \mathbf{1} >$ is a *Sheffer stroke basic algebra*.

Corollary

$< \mathcal{R}^{set}_{Maj}, |, \mathbf{0}, \mathbf{1} >$, $< \mathcal{R}^{set}_{Min}, |, \mathbf{0}, \mathbf{1} >$ and $< \mathcal{R}^{set}_{Con}, |, \mathbf{0}, \mathbf{1} >$ are *Sheffer stroke basic algebras*.

THANK YOU FOR YOUR ATTENTION ☺

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