Some Properties of the Category of Local Sets in Intuitionistic Ramified Type Theory (IRTT)

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Introduction

Introduced in

- Palmgren, *A constructive examination of a Russell-style ramified type theory*, The Bulletin of Symbolic Logic, 24(1), 90-106, 2018

IRTT can be seen as a modern (constructive) reformulation of Russell and Whitehead's type theory in Principia Mathematica. In particular, it contains a version of the infamous reducibility axiom which (when combined with classical logic) collapses the ramified type structure, with respect to extensionality, and reintroduces impredicativity.

However, IRTT can still be shown to be predicative by interpreting it in Martin-Löf type theory.

Introduction

IRTT is an intuitionistic type theory with a ramified power type operation, which also contains a comprehension principle for introducing terms of (ramified) power type. Using "local sets" for such terms, these form a category (as in Bell's *Toposes and local set theories*, from which the terminology is borrowed).

Palmgren expects that this category should give rise to a natural notion of a predicative (elementary) topos. Here we confirm this expectation to the extent that we show that the category of local sets in IRTT is a Π-pretopos with a natural number object.

Introduction

Moreover, extending IRTT with a principle allowing for inductive definitions (due to Palmgren) and a replacement axiom we show that category of local sets in this extended theory is a ΠW-pretopos.

Earlier proposals of the notion of a predicative toposes are the ΠW-pretoposes and stratified pseudotoposes of Moerdijk and Palmgren (2000 respectively 2002), as well as the so-called (strong) predicative toposes of van den Berg (2012), the latter being a ΠW-pretopos satisfing a certain choice principle.

IRTT - Ramified Types

Define the collection $\mathcal{U}$ of ramified type symbols inductively by

- $\mathbf{1}, \mathbf{N} \in \mathcal{U}$,
- If $\mathbf{X}, \mathbf{Y} \in \mathcal{U}$ then $\mathbf{X} \times \mathbf{Y} \in \mathcal{U}$,
- If $\mathbf{X} \in \mathcal{U}$ and $k \in \mathbb{N}$ then $\mathbf{P}_k(\mathbf{X}) \in \mathcal{U}$.

Define the level $\mathrm{lv}(-)$ of a (ramified) type recursively by

$$\begin{aligned} \mathrm{lv}(\mathbf{1}) &= 0, \\ \mathrm{lv}(\mathbf{N}) &= 0, \\ \mathrm{lv}(\mathbf{X} \times \mathbf{Y}) &= \max\big(\mathrm{lv}(\mathbf{X}), \mathrm{lv}(\mathbf{Y})\big), \\ \mathrm{lv}(\mathbf{P}_k(\mathbf{X})) &= \max\big(k+1, \mathrm{lv}(\mathbf{X})\big), \end{aligned}$$

IRTT

- Heyting Arithmetic (HA) in type **N**.
- A unique element $\star : \mathbf{1}$.
- Projection and pairing operations for any (binary) product of types.
- A membership relation $\in : \mathbf{X}, \mathbf{P}_I(\mathbf{X})$ for each type **X** and $I \in \mathbb{N}$.
- Equality $=_\mathbf{X}$ for each type **X** (which is extensional).

IRTT - Terms and Formulas

Formulas and terms of types are defined in a double induction. Formulas will be assigned levels, expressed as $\varphi \in \mathrm{Form}(n)$. Terms are defined as follows:

- as usual, constants and variables are terms, function (symbols) applied to terms are again terms;
- if $\varphi \in \mathrm{Form}(n)$ and $x : \mathbf{X}$ is a variable then $\{x : \mathbf{X} \mid \varphi\} : \mathbf{P}_n(\mathbf{X})$ is a term.

The formulas of level $k \in \mathbb{N}$ are defined as follows

- $\bot \in \mathrm{Form}(k)$.
- If φ, ψ are in Form(k) then so is $\varphi \vee \psi, \varphi \wedge \psi$ and $\varphi \Rightarrow \psi$.
- If $s : \mathbf{X}$ is a term and $X : \mathbf{P}_n(\mathbf{X})$ then $s \in X$ is in Form(k) for $k \geq n$.
- If $s, t : \mathbf{X}$ are terms then $s =_{\mathbf{X}} t \in \mathrm{Form}(k)$ for $k \geq \mathrm{lv}(\mathbf{X})$.
- If $\varphi \in \mathrm{Form}(k)$ and $x : \mathbf{X}$ is a variable, where $k \geq \mathrm{lv}(\mathbf{X})$, then $\forall x : \mathbf{X}.\varphi$ and $\exists x : \mathbf{X}.\varphi$ are in $\mathrm{Form}(k)$.

IRTT - Axioms

A local set is a term of the form $X : \mathbf{P}_l(\mathbf{X})$.

Let a map from $X : \mathbf{P}_l(\mathbf{X})$ to $Y : \mathbf{P}_m(\mathbf{Y})$ be a total functional relation (between X and Y).

Functional Reducibility Axiom
For any $X : \mathbf{P}_l(\mathbf{X})$ and $Y : \mathbf{P}_m(\mathbf{Y})$, if $F : \mathbf{P}_n(\mathbf{X} \times \mathbf{Y})$ is a map, then there is $G : \mathbf{P}_k(\mathbf{X} \times \mathbf{Y})$, where k depends of the l and $\mathbf{Y}$, which is extensionally equal to F:

$$\langle x, y \rangle \in F \iff \langle x, y \rangle \in G.$$

The Category of Local Sets

Definition

The category **LocSet** of local sets has as objects $[X]$, equivalence classes of provably equal terms of the form $X : \mathbf{P}_l(\mathbf{X})$.

A morphism $F : [X] \to [Y]$ is a term $F : \mathbf{P}_k(\mathbf{X} \times \mathbf{Y})$ which defines a map from $X' \in [X]$ to $Y' \in [Y]$ at the particular level specified by the Functional Reducibility Axiom. Two maps F, G are equal if $F = G$ is provable. □

We will try to use as much set-theoretic terminology as possible, and will in general talk as if an object of **LocSet** is simply the term X, but which comes with an implicit (but definite) underlying type (usually the same, boldfaced letter) and level (say, l_X). So, X "is" $X : \mathbf{P}_{l_X}(\mathbf{X})$.

Theorem

LocSet *is a* Π*-pretopos with a natural number object.* □

(A pretopos being an exact, finitely extensive category).

Inductive definitions

To allow for inductively defined constructions we now extend IRTT with a principle due to Palmgren.

Let an inductive definition consist of

- an underlying local set, X
- a local set of rule instances, R;
- a binary relation between rule instances and their conclusion, $C \subseteq X \times R$;
- a binary relation between rule instances and their premises, $P \subseteq X \times R$.

Given such an inductive definition D, we say that $S \subseteq X$ is D-closed if

$$C(x,r) \wedge \forall x' \in X \Big(P(x',r) \Rightarrow x' \in S \Big) \Rightarrow x \in S$$

holds.

Inductive definitions

We now add the following axiom to IRTT,

Principle of General Inductive Definitions (PGID):
For any inductive definition D there is a smallest D-closed local set $M \subseteq X$, where I_M depends on X, R, C and P.

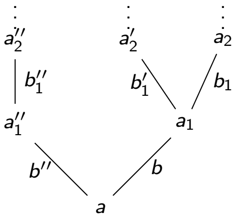
Theorem (Palmgren, unpublished)

IRTT+PGID may be interpreted in Martin-Löf type theory. □

W-types

A Π-pretopos is said to have W-types (or be a ΠW-pretopos) if for any morphism $f : B \to A$ the polynominal functor $P_f = \Sigma_A \Pi_f B^*$ has an initial algebra, where an P_f-algebra for an object X is just a map $P_f(X) \to X$.

In the category of sets, the initial algebra to P_f has underlying set the collection W_f of wellfounded trees indexed by f: nodes labeled by elements of A and a node labeled by a has $B_a := f^{-1}(\{a\})$-many branches going out of it.



$$\begin{aligned} f^{-1}(\{a\}) &= \{b, b''\}, \\ f^{-1}(\{a''_1\}) &= \{b''_1\}, \\ f^{-1}(\{a_1\}) &= \{b'_1, b_1\}, \\ &\vdots \end{aligned}$$

W-types

The algebra is given by the sup-operation, *i.e.* the operation of a pair $\langle a, s\rangle$, where $a \in A$ and $s : B_a \to W_f$, to the tree with root a and tree $s(b)$ attached to a, for each $b \in B_a$.

Set-theoretically, a tree (not necessarily wellfounded) indexed by function $f : B \to A$ can be described as a pair $\langle a, X\rangle$ where $a \in A$ and X is a (certain) set of finite sequences starting at A.

In IRTT we can make similarly construction, obtaining a local set T_F of trees indexed by map $F : B \to A$. The underling type of T_F is then on the form $\mathbf{A} \times \mathbf{P}_q(\mathbf{S})$ for a certain level q, where $\mathbf{S}$ is the type of finite sequences.

PGID can then be applied to obtain a local set W_F of wellfounded trees, for a suitable inductive definition.

W-types

To define a map sup map in IRTT, the intuitive idea is to put, for $a \in A$ and $s : B_a \to W_F$,

$$"\mathrm{sup}(a, s) = (a) \cup \bigcup_{b \in B_a, \alpha \in s(b)} (a) * \alpha" \tag{1}$$

where $*$ means concatenation (here α is a sequence).

Now this involves a quantification over the type of trees, and since this was on the form

$$\mathbf{A} \times \mathbf{P}_q(\mathbf{S})$$

a formula expressing the RHS of (1) will not be of the right level.

The Union-Replacement Axiom

The Union-Replacement axiom of Aczel's CZF is

$$\forall x \in X \exists Y \forall y [y \in Y \leftrightarrow \phi(x, y)] \rightarrow$$
$$\exists U \forall y [y \in U \leftrightarrow \exists x \in X \phi(x, y)]$$

Union-Replacement Axiom (IRTT)
For any $R : \mathbf{P}_r(\mathbf{X} \times \mathbf{Y})$ and $X : \mathbf{P}_l(\mathbf{X})$

$$\forall x \in X \exists Y : \mathbf{P}_q(\mathbf{X}) \forall y : \mathbf{Y}[y \in Y \Leftrightarrow R(x, y)] \Rightarrow$$
$$\exists U : \mathbf{P}_k(\mathbf{X}) \forall y : \mathbf{Y}[y \in U \Leftrightarrow \exists x \in X.R(x, y)]$$

where $k = \mathrm{lv}(\mathbf{X}) \vee l \vee q$.

Theorem

IRTT + Union-Replacement may be interpreted in Martin-Löf type theory.

W-types

With Union-Replacement we can show that the sup operation is a map which constitutes an initial algebra $P_F(W_F) \to W_F$ of P_F.

Theorem

The category of local sets in IRTT+PGID+Union-Replacement is a ΠW-pretopos. □

Future work

Formalization in AGDA.

LocSet may possibly arise through some kind of graded tripos-construction.

Clarify relations to constructive set theories.

Thank you

References

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W-types

Let D be the following inductive definition

- The underlying set to be T_F.
- The set of rules to the collection of pairs $\langle a, s\rangle$ where $a \in A$ and $s : B_a \to T_F$ is a map.
- The conclusions to be the pairs $\langle\langle a, X\rangle, \langle a, s\rangle\rangle$ such that $\langle a, X\rangle$ is the tree obtained by attaching to the node a the tree $s(b)$ with an edge labeled by b, for each $b \in B_a$.
- The premises to
  be the pairs $\langle\langle c, X\rangle, \langle a, s\rangle\rangle$ such that for some $b \in B_a$ we have $s(b) = \langle c, X\rangle$.

By PGID, there is a smallest D-closed subset $W_F \subseteq T_F$.