An extension of a theorem of Zermelo

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- Second order logic is praised for its categoricity results, i.e. its ability to characterize structures.
- But what is second order truth?
- Best understood in terms of provability i.e. truth in all Henkin (rather than “full”) models.
- But Henkin models seem to ruin the categoricity results.
- We show that categoricity can be proved for Henkin models, too, in the form of internal categoricity, which implies full categoricity in full models.

- Zermelo (1930) proved that second order ZFC is κ-categorical for all κ.
- For Henkin models of second order ZFC this is not true in general.

- Let us consider the vocabulary $\{\in_1, \in_2\}$, where both $\in_1$ and $\in_2$ are binary predicate symbols.
- *ZFC*($\in_1$) is the first order Zermelo-Fraenkel axioms of set theory when $\in_1$ is the membership relation and formulas are allowed to contain $\in_2$, too.
- *ZFC*($\in_2$) is the first order Zermelo-Fraenkel axioms of set theory when $\in_2$ is the membership relation and formulas are allowed to contain $\in_1$, too.

Theorem

If $(M, \in_1, \in_2) \models ZFC(\in_1) \cup ZFC(\in_2)$*, then* $(M, \in_1) \cong (M, \in_2)$[1].

[1] Extending Zermelo 1930 and D. Martin “Exploring the Frontiers of Infinity"-paper, draft 2018

- We work in $\mathit{ZFC}(\in_1) \cup \mathit{ZFC}(\in_2)$ in the vocabulary $\{\in_1, \in_2\}$.

- Let $\mathrm{tr}_i(x)$ say that x is transitive in $\in_i$-set theory.
- Let $\mathrm{TC}_i(x)$ be the $\in_i$-transitive closure of x.
- Let $\varphi(x, y)$ be the formula $\exists f \psi(x, y, f)$, where $\psi(x, y, f)$ is the conjunction of the following formulas:

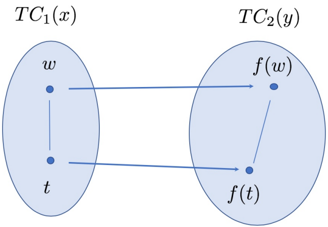
(1) In the sense of $\in_1$, the set f is a function with $\mathrm{TC}_1(x)$ as its domain.

(2) $\forall t \in_1 \mathrm{TC}_1(x)(f(t) \in_2 \mathrm{TC}_2(y))$

(3) $\forall v \in_2 \mathrm{TC}_2(y)\exists t \in_1 \mathrm{TC}_1(x)(v = f(t))$

(4) $\forall t \in_1 \mathrm{TC}_1(x)\forall w \in_1 \mathrm{TC}_1(x)(t \in_1 w \leftrightarrow f(t) \in_2 f(w))$

(5) $f(x) = y$



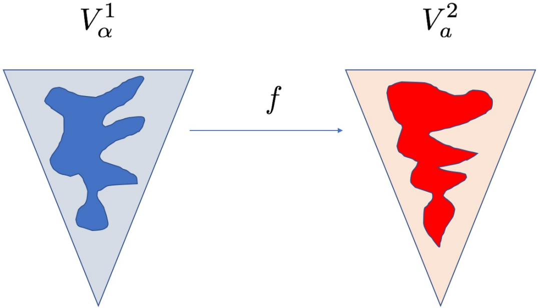
1. If $\psi(x, y, f)$ and $\psi(x, y, f')$, then $f = f'$.
2. If $\varphi(x, y)$ and $\varphi(x, y')$, then $y = y'$.
3. If $\varphi(x, y)$ and $\varphi(x', y)$, then $x = x'$.
4. If $\varphi(x, y)$ and $\varphi(x', y')$, then $x' \in_1 x \leftrightarrow y' \in_2 y$.

- Let $\mathrm{On}_1(x)$ be the $\in_1$-formula saying that x is an ordinal, and similarly $\mathrm{On}_2(x)$.
- For $\mathrm{On}_1(\alpha)$ let V^1_α be the α^{th} level of the cumulative hierarchy in the sense of $\in_1$, and similarly V^2_a.

If $\varphi(\alpha, y)$, then:

1. $\mathrm{On}_1(\alpha)$ if and only if $\mathrm{On}_2(y)$.
2. α is a limit ordinal if and only if y is.

Suppose $\psi(\alpha, y, f)$. If $\mathrm{On}_1(\alpha)$, then there is $\bar{f} \supseteq f$ such that $\psi(V^1_\alpha, V^2_y, \bar{f})$.



Lemma

$\forall x \exists y \varphi(x, y)$ *and* $\forall y \exists x \varphi(x, y)$.

Proof: Consider

$$\forall \alpha(\mathrm{On}_1(\alpha) \to \exists y \varphi(\alpha, y)) \tag{1}$$

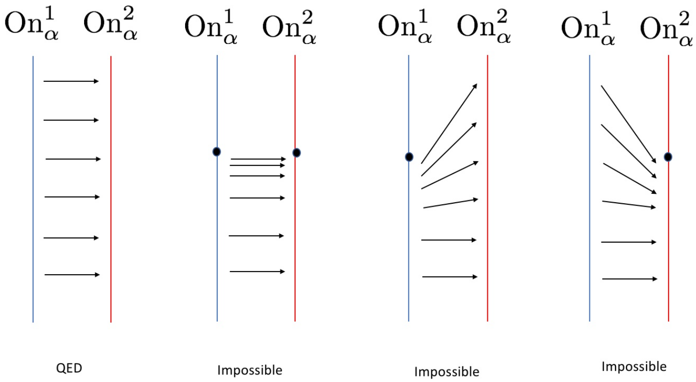
$$\forall y(\mathrm{On}_2(y) \to \exists \alpha \varphi(\alpha, y)). \tag{2}$$

Case 1: (1)$\wedge$(2). The claim can be proved.

Case 2: $\neg$(1)$\wedge\neg$(2). Impossible!

Case 3: (1)$\wedge\neg$(2). Impossible!

Case 4: $\neg$(1)$\wedge$(2). Impossible!


On_α^1 On_α^2
On_α^1 On_α^2
On_α^1 On_α^2
On_α^1 On_α^2
QED
Impossible
Impossible
Impossible

- The class defined by $\varphi(x, y)$ is an isomorphism between the $\in_1$-reduct and the $\in_2$-reduct.
- This concludes the proof.

- Zermelo (1930) showed that if $(M, \in_1)$ and $(M, \in_2)$ both satisfy the **second** order Zermelo-Fraenkel axioms ZFC^2, then $(M, \in_1) \cong (M, \in_2)$.
- Zermelo's result follows from our theorem.
- Note: $ZFC(\in_1)$ and $ZFC(\in_2)$ are first order theories.
- Recall: We allow in these axiom systems formulas from the extended vocabulary $\{\in_1, \in_2\}$.

- Note that $(M, \in_1)$ and $(M, \in_2)$ can be models of $V = L$, $V \neq L$, CH, $\neg CH$, even of $\neg Con(ZF)$.
- It is easy to construct such pairs of models using classical methods of Gödel and Cohen.
- Not all of them can be models of (full) second order set theory ZFC^2.

- An internal categoricity result.
- A strong robustness result for set theory.
- The model cannot be changed “internally”.
- To get non-isomorphic models one has to go “outside” the model.
- But going “outside” raises the potential of an infinite regress of metatheories.

- A similar result holds for *first order* Peano arithmetic: If

$$(M, +_1, \times_1 +_2, \times_2) \models P(+_1, \times_1) \cup P(+_2, \times_2),$$

  then

$$(M, +_1, \times_1) \cong (M, +_2, \times_2).$$

- This extends (and implies) Dedekind's (1888) categoricity result for *second order* Peano axioms.

- Should we think of second order logic or first order set theory as the foundation of classical mathematics?
- The answer: We need a new understanding of the difference between the two. The difference is not as clear as what was previously thought.
- The nice categoricity results of second order logic can be seen already on the first order level, revealing their inherent limitations.

Thank you!