Valued hyperfields, truncated DVRs, and valued fields

Junguk Lee

Institute of mathematics, Wroclaw University

Logic Colloquium 2018
Udine, Italy
23-28, July, 2018

[1] P. Deligne.
Les corps locaux de caractéristique p, limits de corps locaux de caractéristique 0. J.-N. Bernstein, P. Deligne, D. Kazhdan, M.-F. Vigneras, Representations des groupes reductifs sur un corps local, Travaux en cours, Hermann, Paris, 119-157, (1984).

[2] M. Krasner.
Approximation des corps valués complets de caractéristique $p \neq 0$ par ceux caractéristique 0. 1957 Colloque d'algèbre supérieure, tenu á Bruxelles du 19 au 22 décembre.

[3] J. Lee and W. Lee.
On the structure of certain valued fields, preprint.

[4] J. Tolliver.
An equivalence between two approaches to limits of local fields, J. of Number Theory, (2016), 473-492.

$(K, \nu, k, \Gamma, R, \mathfrak{m})$

- (K, ν) is a henselian valued field of mixed characteristic $(0, p)$.
- k is perfect
- (K, ν) is finitely ramified. We have $e_\nu(p)$ is finite, where for each $x \in R$ we define

$$e_\nu(x) := |\{\gamma \in \Gamma | \ 0 < \gamma < \nu(x)\}|.$$

  Note $e_\nu(p)$ is the ramification index. If there is no confusion, we denote e_ν by e.
- $\pi \in R$ is a uniformizer so that Γ is generated by $\nu(\pi)$.
- For each $n > 0$, $R_n := R/\mathfrak{m}^n$, called *the n-th residue ring*.
- If (K, ν) is a discrete complete valued fields, we may assume that ν is normalized, that is, $\nu(p) = 1$. And we still write ν for the unique extension of ν in K^{alg}

Main Goal :
Let K_1 and K_2 be complete discrete valued fields of mixed characteristic with perfect residue fields. There is an positive integer N depending only on the ramification indices such that if **the N-th valued hyper valued fields** of valued fields are **isomorphic over** p, i.e.,

$$\mathcal{H}_N(K_1) \cong_{\mathcal{H}_N(\{p\})} \mathcal{H}_N(K_2)$$

, then K_1 and K_2 are isomorphic, i.e.,

$$K_1 \cong K_2.$$

For each $n > 0$, we define *the n-th valued hyperfield* of K

$$\mathcal{H}_n(K) := (K/(1+\mathfrak{m}^n); +_{\mathcal{H}}, -_{\mathcal{H}}, \cdot_{\mathcal{H}}; 0_{\mathcal{H}}, 1_{\mathcal{H}}; \nu_{\mathcal{H}}),$$

where for $\alpha := [a], \beta := [b] \in K/(1+\mathfrak{m}^n)$,

- $0_{\mathcal{H}} := [0]$ and $1_{\mathcal{H}} := [1]$;
- $\alpha +_{\mathcal{H}} \beta := \{[x+y] |\ x \in a(1+\mathfrak{m}^n), y \in b(1+\mathfrak{m}^n)\}$;
- $\beta := (-_{\mathcal{H}}\alpha)$ is the unique element such that $0_{\mathcal{H}} \in \alpha + \beta$;
- $\alpha \cdot_{\mathcal{H}} \beta := [ab]$; and
- $\nu_{\mathcal{H}}(\alpha) := \nu(a)$,

If there is no confusion, we omit $\mathcal{H}$.

Definition

Let K_1 and K_2 be valued fields. Fix $n > 0$.

1. A map f from $\mathcal{H}_n(K_1)$ to $\mathcal{H}_n(K_2)$ is called *a homomorphism* if for $\alpha, \beta \in \mathcal{H}_n(K_1)$,
   - $f(0) = 0$ and $f(1) = 1$;
   - $f(\alpha\beta) = f(\alpha)f(\beta)$;
   - $f(\alpha + \beta) \subset f(\alpha) + f(\beta)$; and
   - $\nu_1(\alpha) < \nu_1(\beta) \Leftrightarrow \nu_2(f(\alpha)) < \nu_2(f(\beta))$.

   We denote $\mathrm{Hom}(\mathcal{H}_n(K_1), \mathcal{H}_n(K_2))$ for the set of homomorphisms from $\mathcal{H}_n(K_1)$ to $\mathcal{H}_n(K_2)$.
2. A homomorphism f from $\mathcal{H}_n(K_1)$ to $\mathcal{H}_n(K_2)$ is called *over p* if $f([p]) = [p]$. We denote $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_n(K_2))$ for the set of homomorphisms from $\mathcal{H}_n(K_1)$ to $\mathcal{H}_n(K_2)$, which is over p.

Let $(K_1, \nu_1, k_1, \Gamma_1)$ and $(K_2, \nu_2, k_2, \Gamma_2)$ be finitely ramified henselian valued fields of mixed characteristic with perfect residue fields. Let e_1 and e_2 be the ramification indices of K_1 and K_2. Set

$$N = \begin{cases} 1 & \text{if } p \not| e_1 (\text{, tamely ramified}) \\ e_2(p)(1 + e_1^2(p)) + 1 & \text{if } p | e_1 (\text{, wildly ramified}) \end{cases}.$$

Here, $e_1^2(p) = e_1(e_1(p))$ is well-defined because K_1 is of characteristic 0 and $\mathbb{N}$ is a subset of K_1.

Theorem

Suppose (K_1, ν_1) *and* (K_2, ν_2) *be complete discrete valued fields. Then there is a unique lifting map*

$$\mathrm{L}: \ \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_N(K_1), \mathcal{H}_N(K_2)) \to \mathrm{Hom}(K_1, K_2)$$

such that for any homomorphism $\phi : \mathcal{H}_N(K_1) \to \mathcal{H}_N(K_2)$, *set* $g = \mathrm{L}(\phi)$, *and we have*

- *There exists a representaive* β *of* $\phi([\pi_1])$ *which satisfies*

$$\nu_2((g(\pi_1) - \beta) > M(K_1),$$

  where
  $M(K_1) := \max\{\nu_1(\pi_1 - \sigma(\pi_1)) | \ \sigma \in \mathrm{Hom}_{L_1}(K_1, K_1^{\mathrm{alg}}), \ \sigma(\pi_1) \neq \pi_1\}$
  and $L_1 (\subset K_1)$ *is the quotient field of the Witt ring* $W(k_1)$.
- $\phi_{red,1} \circ \mathcal{H}_1 = \mathcal{H}_1 \circ g$, *where* $\phi_{red,1} : \mathcal{H}_1(K_1) \to \mathcal{H}_1(K_2)$ *is the natural reduction map of* ϕ *and* $\mathcal{H}_1$ *is the natural projection map to the first valued hyperfield.*

Main Theorem 1

$$\begin{array}{ccc} K_1 & \xrightarrow{\quad g \quad} & K_2 \\ \downarrow{\scriptstyle \mathcal{H}_1} & & \downarrow{\scriptstyle \mathcal{H}_1} \\ \mathcal{H}_1(K_1) & \xrightarrow{\phi_{red,1}} & \mathcal{H}_1(K_2) \end{array}$$

Example

Let $K_1 = \mathbb{Q}_3(\sqrt{3})$ and $K_2 = \mathbb{Q}_3(\sqrt{-3})$. Since $e_1 = e_2 = 2$, $N = 1$. Set $\pi_1 = \sqrt{3}$ and $\pi_2 = \sqrt{-3}$. Note that the sets of Teichmüller representatives of K_1 and K_2 are $(S :=) S_1 = S_2 = \{-1, 0, 1\}$. Consider a homomorphism $f \in \mathrm{Hom}(\mathcal{H}_1(K_1), \mathcal{H}_1(K_2))$ by sending $[\sqrt{3}] \mapsto [\sqrt{-3}]$. Note that for each $a \in S$, $f([a]) = [a]$. We have that

$$f : \mathcal{H}_1(K_1) \cong \mathcal{H}_1(K_2), [\sqrt{3}] \mapsto [\sqrt{-3}]$$

which is *not* over p. But K_1 and K_2 are never isomorphic.

Theorem

Let K_1 and K_2 be finitely ramified henselian valued fields of mixed characteristic $(0, p)$ with perfect residue fields. The followings are equivalent:

1. $K_1 \equiv K_2$.
2. $\mathcal{H}_N(K_1) \equiv_{\mathcal{H}_N(\{p\})} \mathcal{H}_N(K_1)$.

Furthermore, if we take $N' = e_2(p)(1 + e_1^2(p)) + 1$,

3. *(S. Basarab, L. and W.Lee)* $R_{1,N'} \equiv R_{2,N'}$ *and* $\Gamma_1 \equiv \Gamma_2$.

Example

Let $K_1 = \mathbb{Q}_3(\sqrt{3})$ and $K_2 = \mathbb{Q}_3(\sqrt{-3})$. Then $e_1 = e_2 = 2$, and $N = 1$ and $N' = 3$. We have that $K_1 \not\equiv K_2$ by the sentence $\exists X(X^2 + 3 =)0$, and so $\mathcal{H}_1(K_1) \not\equiv_{\mathcal{H}_1(\{3\})} \mathcal{H}_1(K_2)$ and $R_{1,3} \not\equiv R_{2,3}$. But $R_{1,1} = R_{2,1} = \mathbb{F}_3$.

Grazie!