Learning Equivalence Structures

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joint work with Ekaterina Fokina and Timo Koetzing

Computational Learning Theory

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More recently researchers applied the machinery of CLT to algebraic structures:

- (Stephan, Ventsov, 2001): Learning ring ideals of commutative rings.
- (Harizanov, Stephan, 2002): Learning subspaces of V_∞.

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- **InfEx**$_{\cong}(\mathbb{K})$ denotes the class of families of $\mathbb{K}$-structures that are **InfEx**$_{\cong}$-learnable.

Our framework, continued

Our notation comes from classical CLT: **Inf** (short for *informant*) means that we receive both positive and negative information about $\mathcal{S}$; **Ex** (short for *explanatory*) means that **M** shall converge on a single input for $\mathcal{S}$.

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Remark: our model shares many analogies with the First-order Framework introduced in (Martin, Osherson, 1998).

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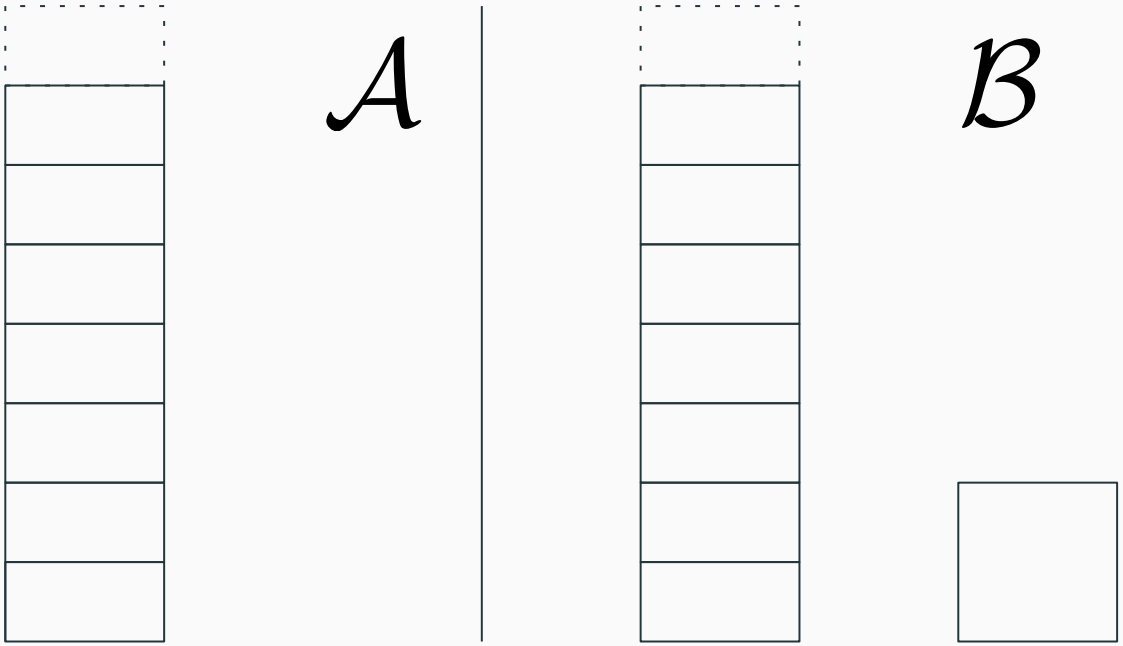
- (Downey, Melnikov, Ng, 2016): there is a Friedberg enumeration of computable equivalence structures.
- A non-Friedberg one is of course much more easy to be defined, e.g., for all n, let the size of the $\mathcal{E}_n$-classes be the cardinality of the columns of W_n.

Example of learnability, 1



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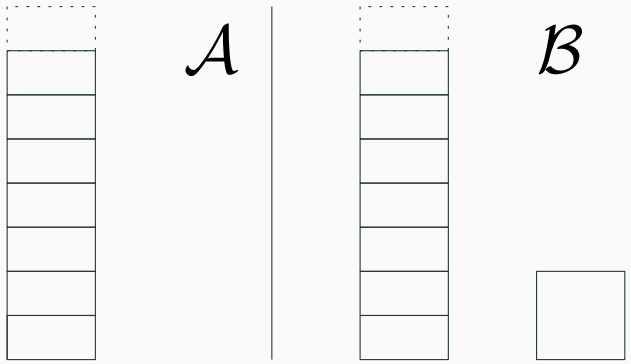

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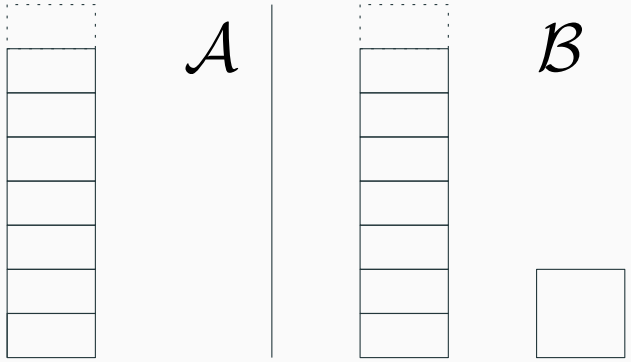

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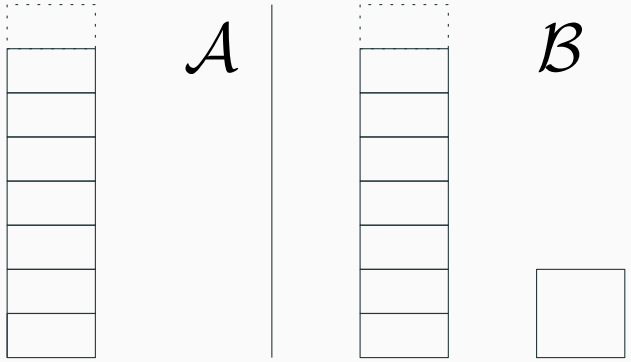

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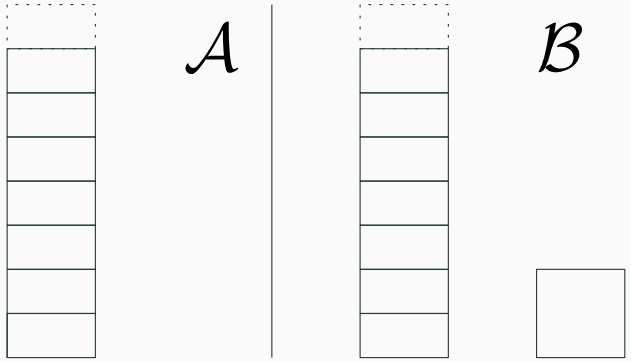

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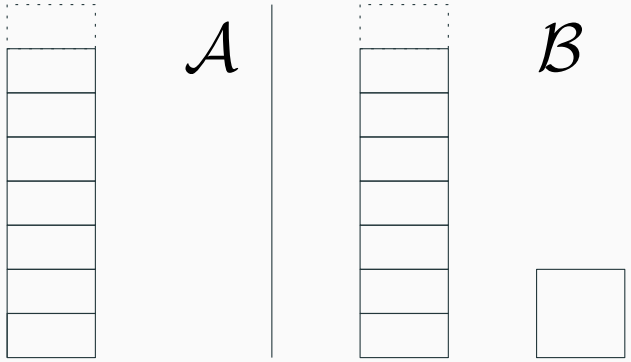

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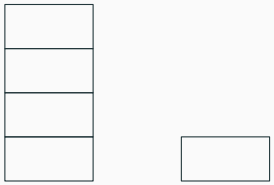

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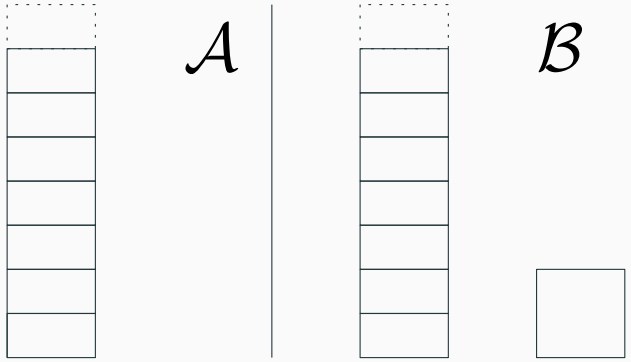

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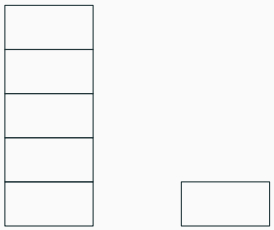

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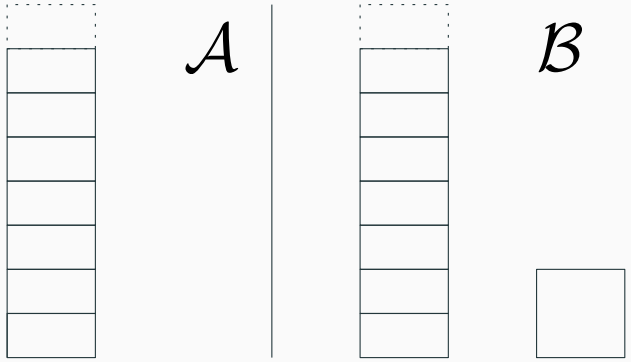

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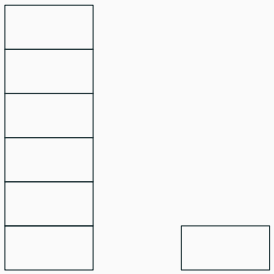

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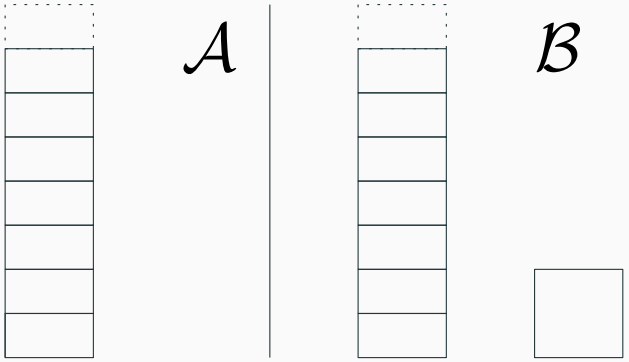

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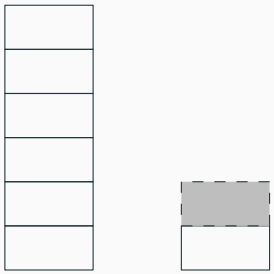

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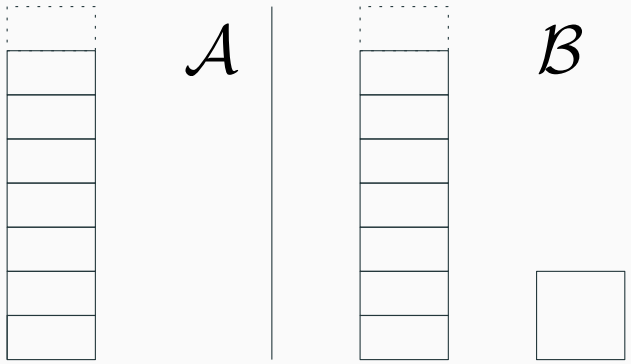

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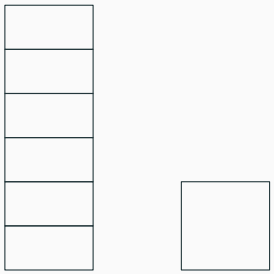

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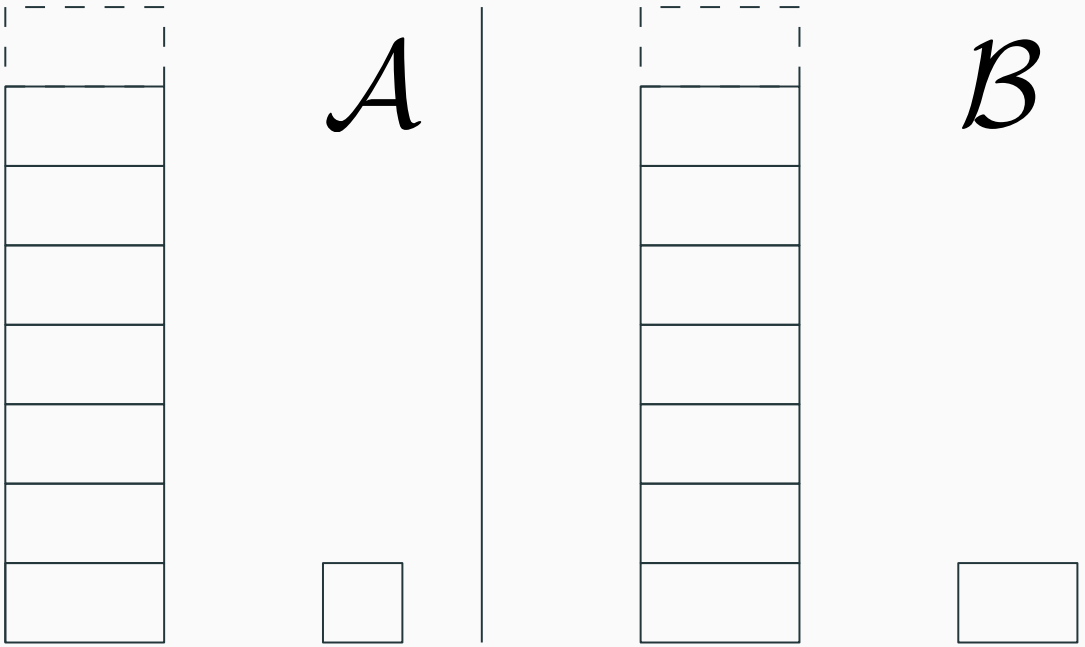

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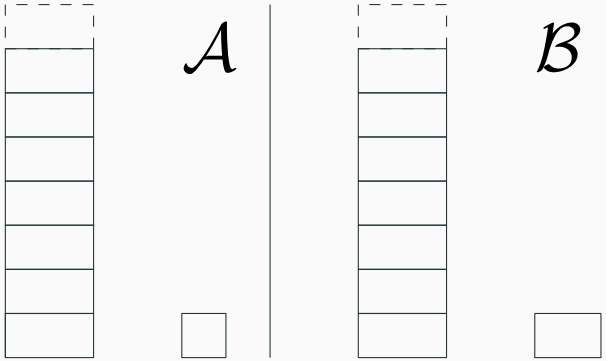

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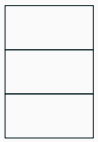

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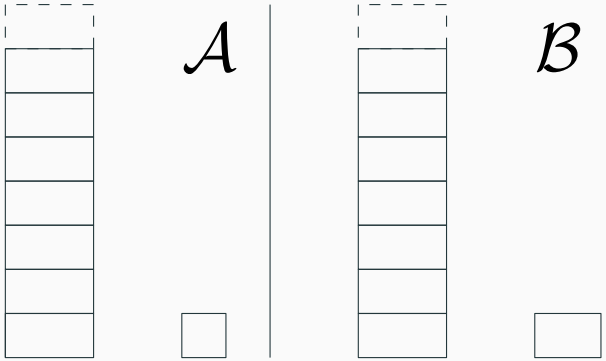

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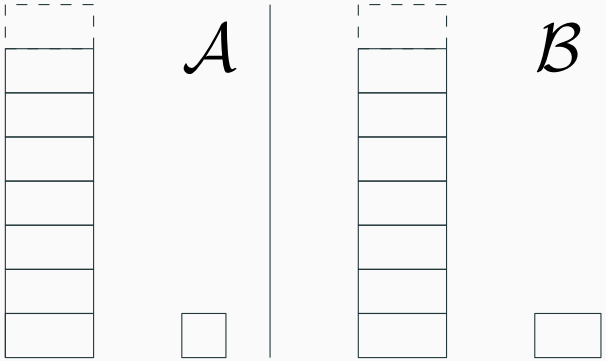

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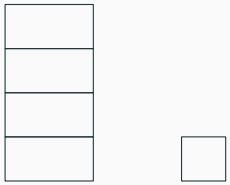

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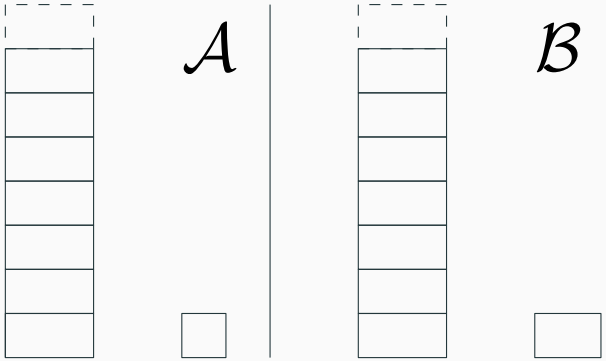

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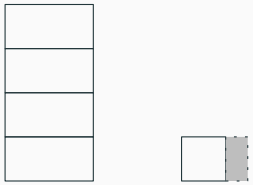

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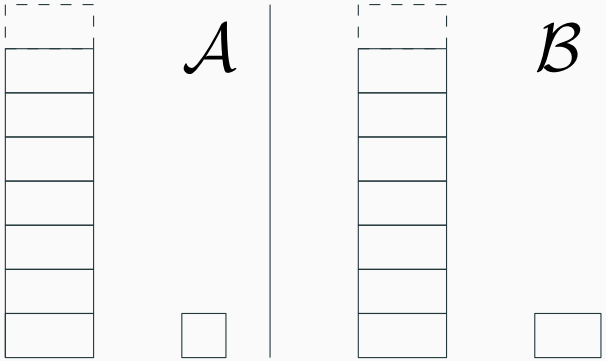

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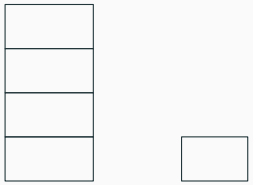

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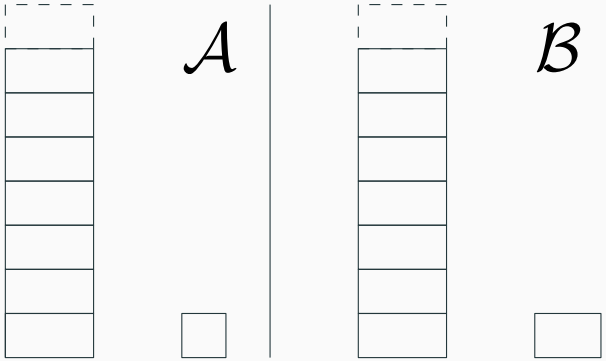

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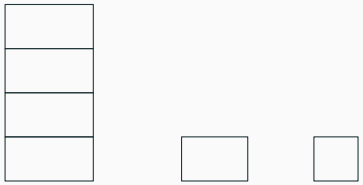

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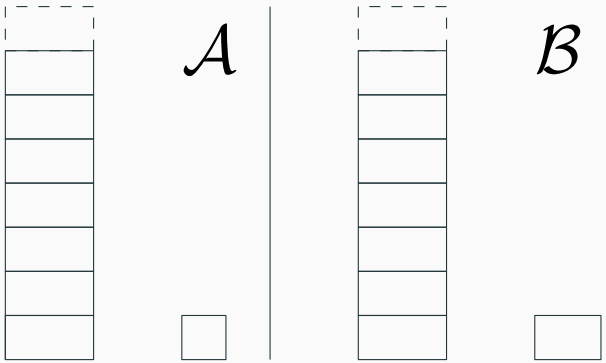

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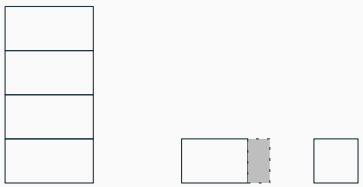

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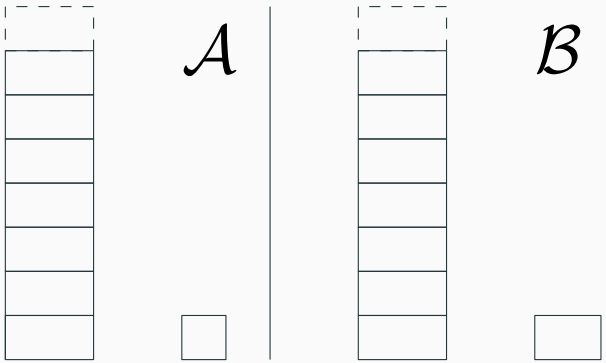

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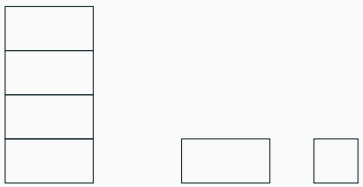

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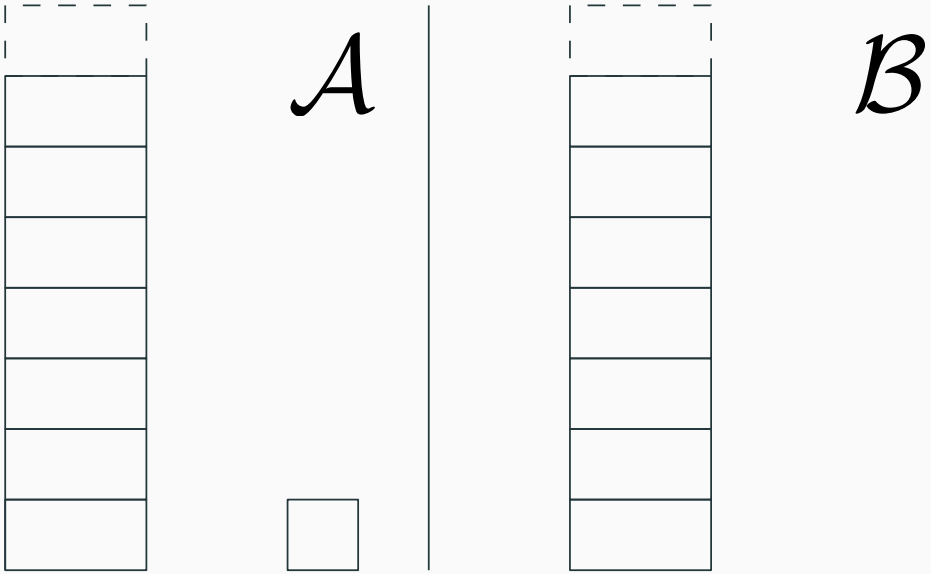

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Strategy

- Assume that **M** learns $\{\mathcal{A}, \mathcal{B}\}$,
- Construct by stages a structure

$$\mathcal{S} \in \{\mathcal{T} : \mathcal{T} \cong \mathcal{A} \vee \mathcal{T} \cong \mathcal{B}\}$$

such that **M** fails to learn $\mathcal{S}$.

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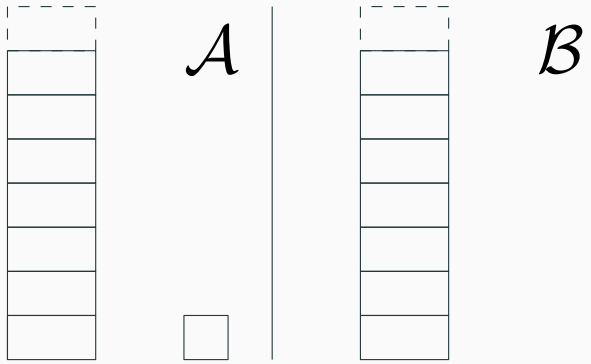

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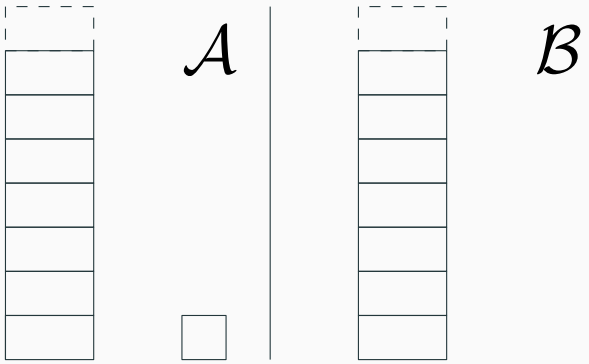

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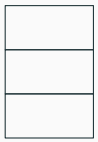

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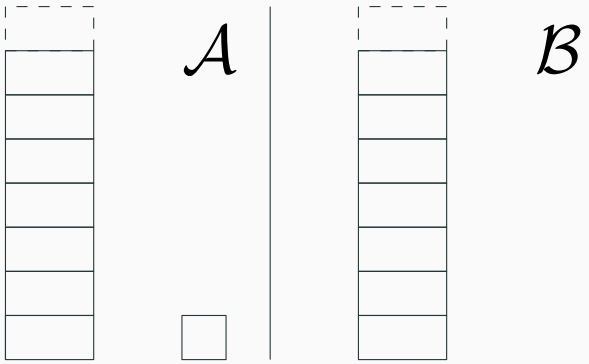

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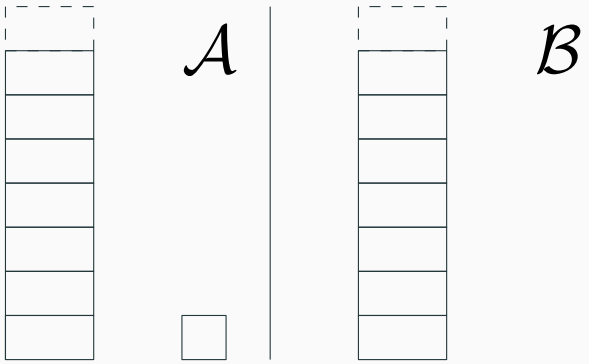

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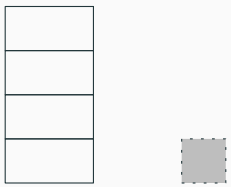

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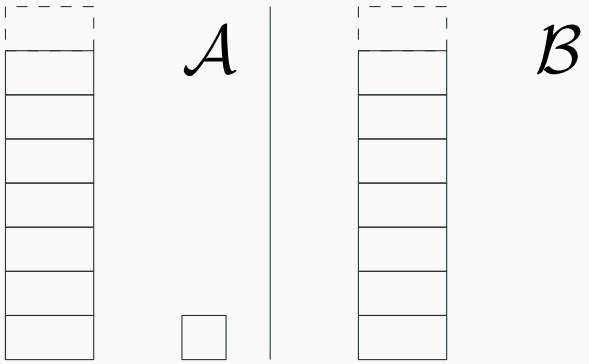

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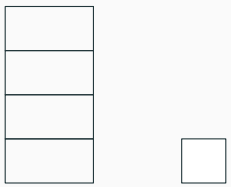

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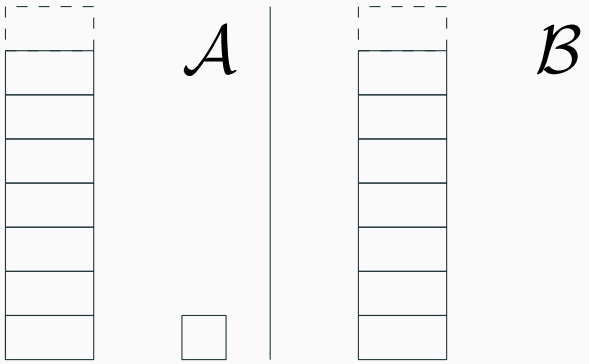

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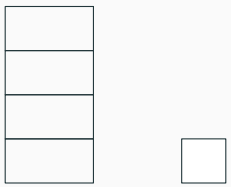

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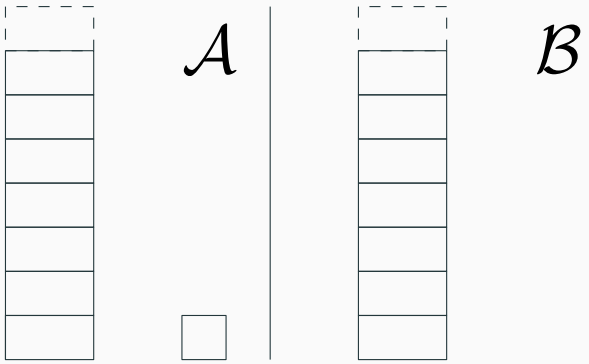

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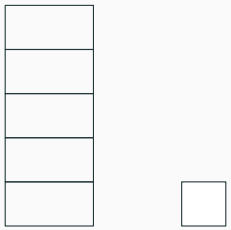

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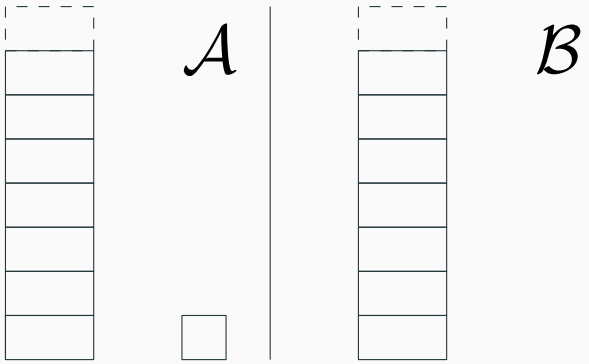

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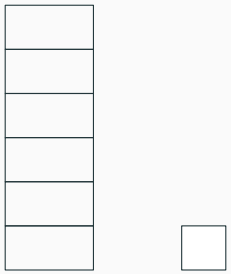

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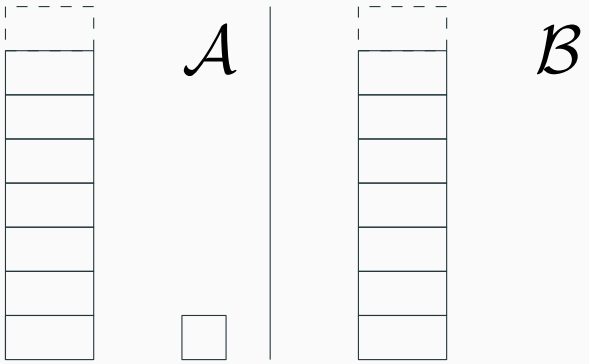

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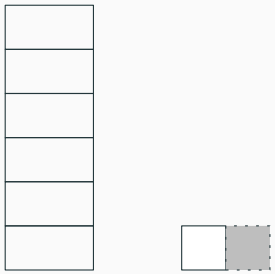

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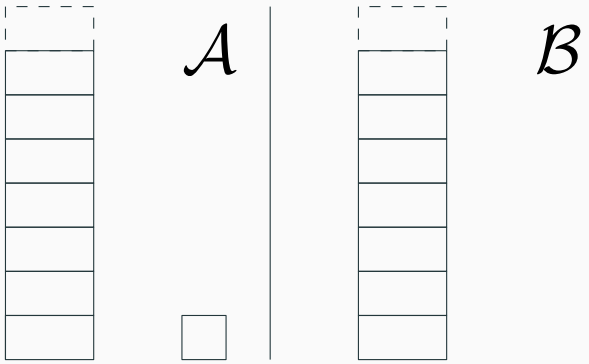

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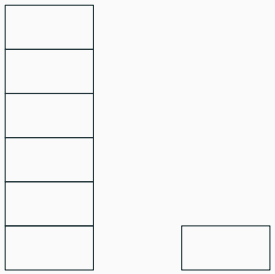

Example of nonlearnability, 2

Recall that the character $c_{\mathcal{A}}$ of $\mathcal{A}$ is

$$c_{\mathcal{A}} = \{\langle k, i\rangle : \mathcal{A} \text{ has } i \text{ equivalence classes of size } k\}.$$

Define $\mathfrak{A} = \{\mathcal{A}_i\}_{i\in\omega^+}$ such that, for all $\mathcal{A}_i$'s, $c_{\mathcal{A}_i} = \{\langle k, 1\rangle : k \neq i\}$.

Proposition

$\mathfrak{A} \notin \mathbf{InfEx}_{\cong}(\mathbb{E})$.

Finite separability

- $\mathcal{S}$ is a limit of a finite family $\mathfrak{A}$ if there is $\mathcal{A} \in \mathfrak{A}$ such that

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- $\mathfrak{A}$ is finitely separable if, for all $\mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{B}$ has no limits in $\mathfrak{A}$.

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Theorem

If $\mathfrak{A}$ is a family of equivalence structures, then

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2. *There is $\mathfrak{A} \notin \mathbf{InfEx}_{\cong}(\mathbb{E})$ such that, for all $\mathfrak{B} \subseteq \mathfrak{A}$, if $\mathfrak{B}/_{\cong}$ is finite, then $\mathfrak{B} \in \mathbf{InfEx}_{\cong}(\mathbb{E})$.*

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$\mathfrak{A} \in \mathbf{0}^{(\alpha)}$-$\mathbf{InfEx}_{\cong}(\mathbb{K})$ if there is a Turing functional Ψ_e such that $\Psi_e^{(\mathbf{0}^{(\alpha)} \oplus \mathbf{d})}$ $\mathbf{InfEx}_{\cong}$-learns $\mathfrak{A}$, for all $\mathbf{d}$ that enumerates $\mathfrak{A}$.

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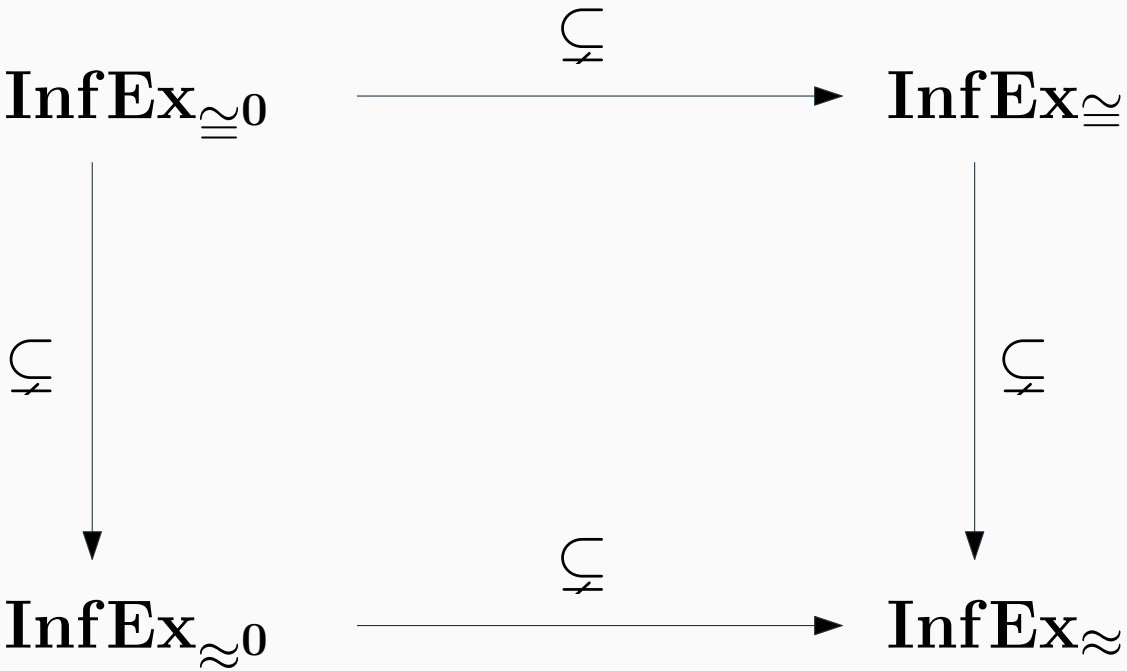
3. $\mathbf{Ex} \mapsto \mathbf{BC}$: in **BC**-learning (short for *behaviourally correct*) the learner is allowed to change its mind infinitely many times as far as almost all its conjectures lie in the same E-class (with E defined as in 2.).

Other learnability classes, II



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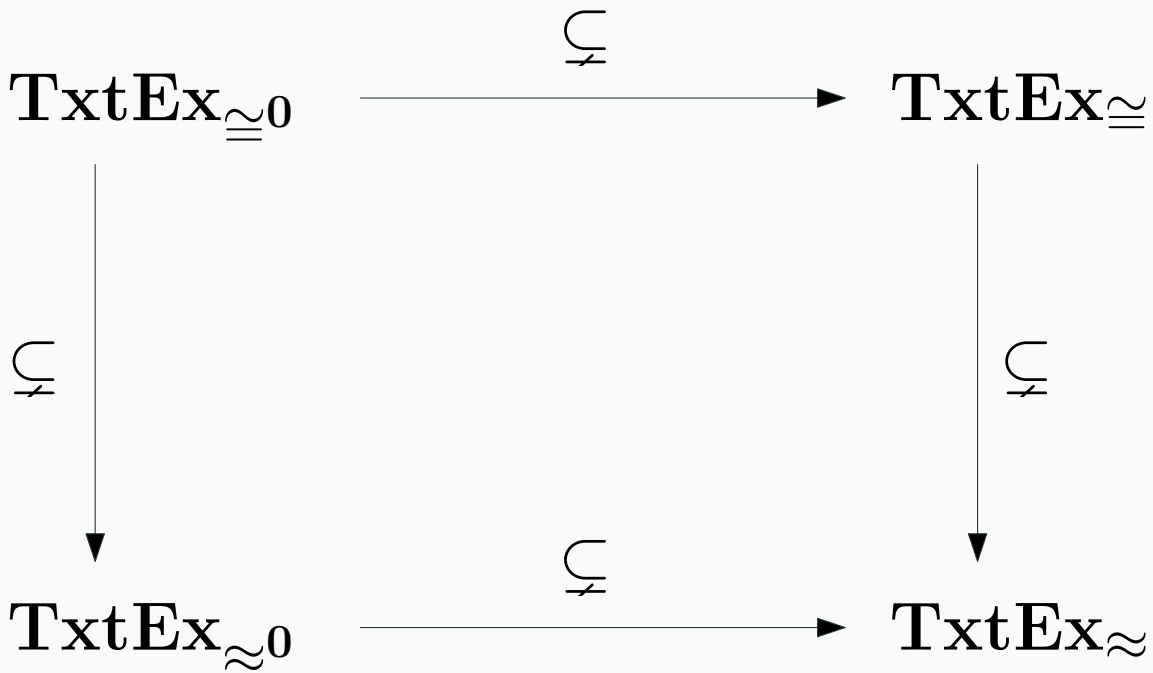

Further directions

1. Apply the framework to other classes $\mathbb{K}$.
2. Investigate **BC**-learnability.

Thank you!