Gödel's Koan and Gentzen's Second Consistency Proof

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Gödel's Koan

"A koan is a story, dialogue, question, or statement, which is used in Zen practice to provoke the "great doubt" and test a student's progress in Zen practice." (Wikipedia)

Problem 26 - TLCA list of open problems
Submitted by Henk Barendregt
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Statement: Assign (in an 'easy' way) ordinals to terms of the simply typed lambda calculus such that reduction of the term yields a smaller ordinal.
Problem Origin: First posed by Kurt Gödel.
Construct an easy assignment of (possibly transfinite) ordinals to terms of the simply typed lambda calculus, i.e., a map
$F : \Lambda_{\rightarrow} \Rightarrow \{\alpha : \alpha \text{ is an ordinal}\}$ such that

$$\forall M, N \in \Lambda_{\rightarrow} [\mathsf{M} \rightarrow_{\beta} \mathsf{N} \Rightarrow \mathsf{F[M]} < \mathsf{F[N]}].$$

“As the problem is formulated it contains an element of vagueness as it is presented as the problem of finding a simple or easy ordinal assignment for strong normalization of the beta-reduction of simply typed lambda calculus. Whether a proof is sufficiently easy to categorize as a solution is thus a matter of opinion.” (Annika Kanckos, *Logic Colloquium*, Stockholm, 2017)

Previous Results

- Howard [1968]
- de Vrijer [1987]
- Durante [1999]
- Beckmann [2001]
- Sanz [2006]

And quite recently, Annika Kanckos [2017]

Some possible solutions

- Worst reduction sequences
- $* - derivations$ (*disastrous derivations*)
- Minp-graphs (Cruz, Haeusler, and Gordeev)
- Gentzen reductions (Pereira and Haeusler)

Worst reduction sequences

1. Define the concept of *worst reduction sequence*
2. Prove that, for any derivation Π, the worst reduction sequence for Π is finite.
3. Define $lp[\Pi]$ as the length of the worst reduction sequence for Π.
4. Define for any derivation Π the measure $on[\Pi]$ as: $on[\Pi] = lp[\Pi]$.
5. Show that if Π reduces to Π', then $on[\Pi'] < on[\Pi]$.

Problem: The measure *on* depends on a normalization strategy

General description of the method

Main idea - to associate to a given derivation Π a derivation Π^* such that all possible maximum formulas that may arise in reduction sequences starting with Π occur in Π^*.

1. Applications of $\rightarrow -Int$ that do not discharge assumptions produce "vacuous reductions", while those that discharge assumptions produce "multiplicative reductions".
2. A derivation that can only produce "vacuous reductions" is called $* - derivations$.
3. Any reduction applied to a $* - derivation$ produces a decrease in its length.
4. Define a method to check if a derivation is a $* - derivation$ (this is done by means of $\alpha - segments$.

5. Associate to a given derivation Π a $* - derivation$ Π^*. This derivation Π^* will contain all possible maximum formulas of Π [occurring as pair of formula occurrences, the α pairs].
6. We can now "count" the number of such pairs (of all possible maximum formulas of Π).
7. This number will be the natural ordinal of Π, $on(\Pi)$.
8. Clearly, if Π reduces to Π', then $on(\Pi') < on(\Pi)$

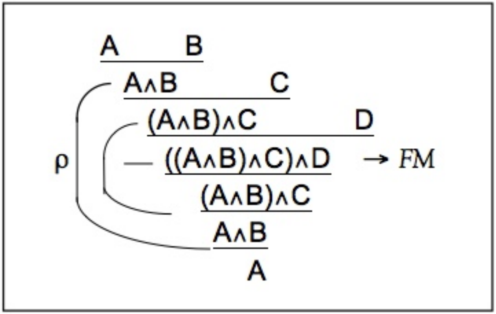
Basic definions

1. Definition: A derivation Π is said to be a *star-derivation* iff $\forall \Pi'$ such that Π reduces to Π', $l(\Pi') < l(\Pi)$.
2. The notion of $\alpha - segment$ will allow us to discover whether a derivation Π is a star derivation or not.

3. A segment in a derivation Π is a sequence $A_1, A_2, ..., A_n$ of consecutive formula occurrences in a thread in Π.
4. Let S be the segment $A_1, ..., A_n$. The *center of S*, denoted by c(S), is the rational number given by: c(S) $= (n+1)/2$. If c(S) is an integer, then $A_c(S)$ is called the *central occurrence of S*.
5. Let S $= A_1, ..., A_n$ be a segment in a derivation Π of central occurrence A_i. We say that S is an $\alpha - segment$ of level 1 in Π if, for all j such that $1 \leq j \leq i = c(S)$, A_j (an occurrence in the first half of S) and A_{n-j+1} (an occurrence of the same formula in the second half of S symmetric to A_j), where A_j is the consequence of an introduction rule and A_{n-j+1} is the major premise of an elimination rule..

$\alpha - segment$ of level 1

Consider the following derivation:



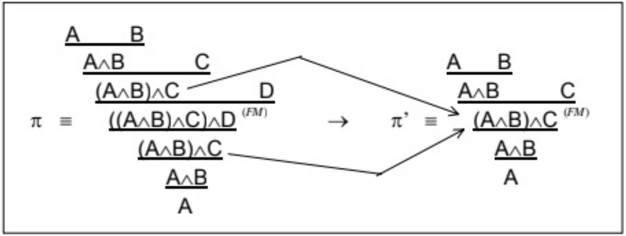
$\alpha - segment$ of level 1

The segment

$(A \wedge B), ((A \wedge B) \wedge C), (((A \wedge B) \wedge C) \wedge D), ((A \wedge B) \wedge C), (A \wedge B)$

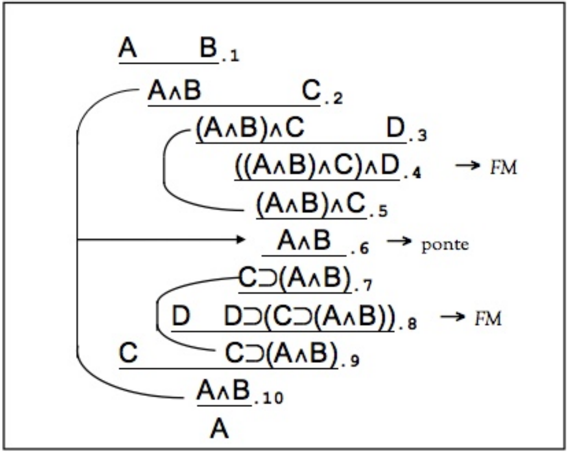
is an $\alpha - segment$ of level 1. The formula $((A \wedge B) \wedge C)$ is a candidate to be a maximum formula.

$\alpha - segment$



$\alpha - segment$ of arbitrary level

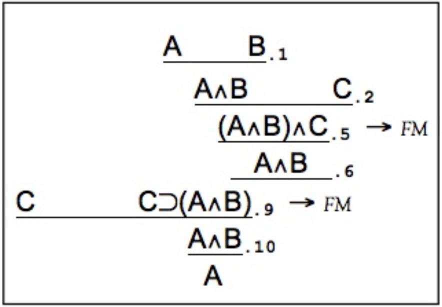
Consider the following derivation:



An $\alpha - segment$ of level 2

$S = (A \wedge B)_2, ((A \wedge B) \wedge C)_3, (((A \wedge B) \wedge C) \wedge D)_4, ((A \wedge B) \wedge C)_5,$
$(A \wedge B)_6, (C \rightarrow (A \wedge B))_7, (D \rightarrow (C \rightarrow (A \wedge B)))_8,$
$(C \rightarrow (A \wedge B))_9, (A \wedge B)_{10}.$

This derivation reduces to:



The moral is: sequences of segments separated by occurrences of the same formula may also determine pairs of formula occurrences that reductions may turn into maximum formulas!

Definition

We say that an occurrence of a formula A in a derivation Π is *heavy* in Π iff A is the major premiss of an elimination rule and belongs to some $\alpha - segment$ in Π.

Remark: If A is heavy in Π and is not a maximum formula in Π, then A is a candidate to be a maximum formula in Π.

Non-multiplicative reductions

Some $\rightarrow -reductions$ reduces the size of derivations. These are called *non-multiplicative reductions*.

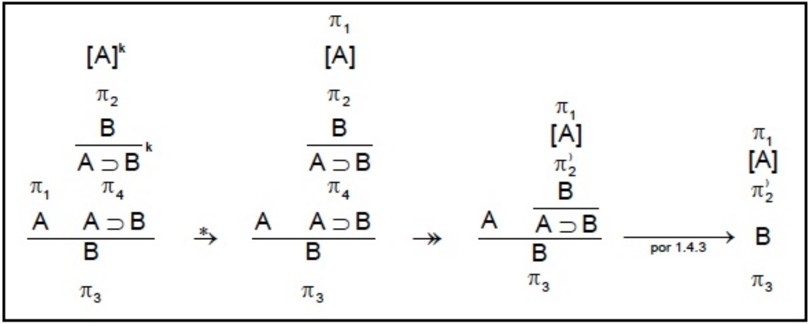
$$\dfrac{\begin{matrix}\Pi_1 \\ A\end{matrix} \qquad \dfrac{\begin{matrix}\Pi_2 \\ B\end{matrix}}{(A \rightarrow B)}}{\begin{matrix}B \\ \Pi_3\end{matrix}}$$

Reduces to:

$$\begin{matrix}\Pi_2 \\ [B] \\ \Pi_3\end{matrix}$$

Multiplicative occurrences

Consider the following derivation:



Definition

A derivation Π is said to be a *star-derivation* iff $\forall \Pi'$ such that Π reduces to Π', $l(\Pi') < l(\Pi)$.

Theorem

Let Π *be a derivation in* $I_{\rightarrow}$. *Then, there is a unique* $* - derivation$ Π^{*}*associated to* Π.

Definition

Let Π be a derivation in $I_{\rightarrow}$. The *weight* of Π, $w(\Pi)$, is defined as the number of heavy formula occurrences in Π.

Definition

Let Π be a derivation in $I_{\rightarrow}$ and let Π^* be the unique $* - derivation$ associated to Π. The *natural ordinal* of Π, $no(\Pi)$, is defined as: $on(\Pi) = w(\Pi^*)$

Theorem

Let Π *be a derivation in* $I_{\rightarrow}$. *If* Π *reduces to* Π', *then* $on(\Pi') < on(\Pi)$.

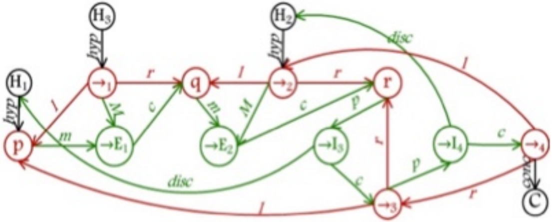
Minp-graphs

Main idea: to use *proof-graphs* to represent proofs in the implicational fragment of minimal logic!

Minp-graphs

$$\dfrac{\dfrac{\dfrac{\dfrac{[p]^1 \quad p \to q}{q} \to\text{-}E \qquad [q \to r]^2}{r} \to\text{-}E}{p \to r}\,(\to\text{-}I,1)}{(q \to r) \to (p \to r)}\,(\to\text{-}I,2)$$

$\Downarrow$ *trans*



Theorem

Every standard tree-like natural deduction Π in the implicational fragment of minimal logic has a unique (up to graph-isomorphism) [F-minimal] Mimp-like representation $G[\Pi]$

Definition

Let G be a Minp-graph. We define $Nmax(G)$ as the number of maximal formulas in G.

Theorem

Let G be a Minp-graph. If G reduces to G', then $Nmax(G') < Nmax(G)$.

Important: The measure does not depend on any normalization strategy

Gentzen's second (published) consistency proof 1938 - The *New Version*

The *New Version*

Gentzen's *New Version* can be roughly described as follows:

1. Consider an LK like formulation of arithmetic.
2. Define an assignment Ord of ordinals $< \epsilon_0$ to proofs in the system.
3. Define a set of reduction operations OP.
4. Show that if there is a proof Π in the system of the empty sequent "$\Rightarrow$", then there is always an operation $op \in OP$ such $op[\Pi]$ is a proof of "$\Rightarrow$" and $Ord[op[\Pi]] < Ord[\Pi]$.
5. The result immediately follows by transfinite induction up to ϵ_0.

The *New Version* and cut-elimination

In 1982 it was showed how the techniques and methodology used by Gentzen in NV could be used to obtain cut-elimination results for sequent calculi for classical first order logic (LK) and for intuitionistic first order logic (LJ). In fact, it was shown, somewhat surprisingly, how the assignment used by Gentzen produces an interesting measure for the estimation of the length of normal proofs in these calculi. Prawitz proved the same result for Natural Deduction in 2015.

More interesting: we can use the reductions of the *New Version* to obtain a *strong cut-elimination* result for the propositional part of LK.

Operational Reductions

Let Π Be:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \Gamma_1 \Rightarrow \Delta_1, A\end{matrix} \qquad \begin{matrix}\Pi_2 \\ \Gamma_1 \Rightarrow \Delta_1, B\end{matrix}}{\Gamma_1 \Rightarrow \Delta_1, A \wedge B}}{\begin{matrix}\Pi_3 \\ \Gamma \Rightarrow \Delta, A \wedge B\end{matrix}} \qquad \dfrac{\dfrac{\begin{matrix}\Sigma_1 \\ A, \Theta_1 \Rightarrow \Psi_1\end{matrix}}{A \wedge B, \Theta_1 \Rightarrow \Psi_1}}{\begin{matrix}\Sigma_2 \\ A \wedge B, \Theta \Rightarrow \Psi\end{matrix}}}{\dfrac{\begin{matrix}\Gamma, \Theta \Rightarrow \Delta, \Psi \\ \Sigma_3\end{matrix}}{\begin{matrix}\Gamma_3 \Rightarrow \Theta_3 \\ \Sigma_4\end{matrix}}}$$

The derivation Π reduces to the following derivation Π':

$$
\dfrac{
\begin{array}{c}
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1 \Rightarrow \Delta_1, A\end{array}}{\Gamma_1 \Rightarrow \Delta_1, A, A \wedge B}\\
\Pi_3'\\
\Gamma \Rightarrow \Delta, A, A \wedge B
\end{array}
\quad
\begin{array}{c}
\dfrac{\begin{array}{c}\Sigma_1\\ A, \Theta_1 \Rightarrow \Psi_1\end{array}}{A \wedge B, \Theta_1 \Rightarrow \Psi_1}\\
\Sigma_2\\
A \wedge B, \Theta \Rightarrow \Psi
\end{array}
}{\Gamma, \Theta \Rightarrow \Delta, A, \Psi}\\
\Sigma_3''\\
\Gamma_3 \Rightarrow \Theta_3, A
\end{array}
\quad
\begin{array}{c}
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1 \Rightarrow \Delta_1, A\end{array} \quad \begin{array}{c}\Pi_2\\ \Gamma_1 \Rightarrow \Delta_1, B\end{array}}{\Gamma_1 \Rightarrow \Delta_1, A \wedge B}\\
\Pi_3\\
\Gamma \Rightarrow \Delta, A \wedge B
\end{array}
\quad
\begin{array}{c}
\dfrac{\begin{array}{c}\Sigma_1\\ A, \Theta_1 \Rightarrow \Psi_1\end{array}}{A \wedge B, A, \Theta_1 \Rightarrow \Psi_1}\\
\Sigma_2'\\
A \wedge B, A, \Theta \Rightarrow \Psi
\end{array}
}{\Gamma, A, \Theta \Rightarrow \Delta, \Psi}\\
\Sigma_3'\\
A, \Gamma_3 \Rightarrow \Theta_3
\end{array}
}{\Gamma_3, \Gamma_3 \Rightarrow \Theta_3, \Theta_3}
$$

$$
\begin{array}{c}
\vdots\\
\Gamma_3 \Rightarrow \Theta_3\\
\Sigma_4
\end{array}
$$

Theorem

(Strong Cut-Elimination): Every reduction sequence for a derivation Π in **LKP** *is finite.*

Proof.

By induction over the value $G_3(\Pi)$ associated with a derivation Π. We show that if Π reduces to Π' then, $G_3(\Pi) < G_3(\Pi')$. □

Can we extend this strong cut-elimination result to the implicational fragment of LJ? No, not directly!!

Gentzen's reduction takes us out of LJ!!

Possible solution:

The system FIL

$$\text{Axiom}\ \frac{}{A(n) \vdash A\{n\}} \qquad \frac{\Gamma \vdash A/S, \Delta' \qquad A(n), \Gamma' \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta', \Delta^*}\ \text{Cut}$$

$$Ex_L\ \frac{\Gamma, A(n), B(m), \Gamma' \vdash \Delta}{\Gamma, B(m), A(n), \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash A/S, B/S', \Delta}{\Gamma \vdash B/S', A/S, \Delta}\ Ex_R$$

$$W_L\ \frac{\Gamma \vdash \Delta}{\Gamma, A(n) \vdash \Delta^*} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A/\{\ \}, \Delta}\ W_R$$

$$Con_L\ \frac{\Gamma, A(n), A(m) \vdash \Delta}{\Gamma, A(k) \vdash \Delta^*} \qquad \frac{\Gamma \vdash A/S, A/S' \Delta}{\Gamma \vdash A/S \cup S', \Delta}\ Com_R$$

$$\to_L\ \frac{\Gamma \vdash A/S, \Delta \qquad \Gamma', B(n) \vdash \Delta'}{\Gamma, \Gamma', A \to b(n) \vdash \Delta, \Delta'} \qquad \frac{\Gamma, A(n) \vdash B/S, \Delta}{\Gamma \vdash (A \to B)/S - \{n\}, \Delta}\ \to_R$$

Similar problems appear in FIL: reductions may take us out of FIL!!

Solution: define *FIL-notations*

Conclusions

- Minp-graphs appear as an interesting possibility - the measure does not depend on specific normalization strategies!
- The $* - derivation$ strategy does not depend on a normalization strategy, but the method used to associate a $* - derivation$ Π^* to a derivation Π is too close to a normalization strategy.
- We can define FIL-notations (analogous to Scarpellini's "almost intuitionistic derivations").
- We can define an (natural number) assignment G that establishes the desired result for FIL-notations.
- We can map FIL-derivations into FIL-notations.