Elementary theories and hereditary undecidability for semilattices of numberings

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Outline

Definition

Any surjective mapping α of the set ω of natural numbers onto a nonempty set A is called a *numbering* of A.

- If α is 1-1 , then it is usually called Friedberg numberings.
- Let $\theta_\alpha \leftrightharpoons \{< x, y > |\alpha x = \alpha y\}$. A numbering α is called decidable (positive) if θ_α is computable. (computably enumerable).

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Let α and β be numberings of A. We say that a numbering α is *reducible* to a numbering β (in symbols, $\alpha \leqslant \beta$) if there exists a computable function f such that $\alpha(n) = \beta(f(n))$ for any $n \in \omega$.

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- Define some language L and the interpretation of that language determined as a partial surjective mapping $i : L \to A$. For any object $a \in A$, each "formula"in $i^{-1}(a)$ is interpreted as a description of a.
- For example, if A consists of partial computable functions then $i^{-1}(a)$ may be considered as a set of programs of Turing machines for a.
- If A is a set of c.e. sets then $a \in A$ is definable by Σ^0_1 -formulas in arithmetics and we could consider $i^{-1}(a)$ as a collection of such formulas.
- For L, we consider a *Godel* numbering $G : \omega \to L$.

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A numbering $\alpha : \omega \to A$ is called a computable numbering of A in the language L with respect to the interpretation i if there exists a computable function f for which the formula $G(f(n))$ distinguishes an element $\alpha(n)$ in L relative to i, i.e. $\alpha(n) = i(G(f(n)))$ for all $n \in \omega$.

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Numbering $\alpha : \omega \mapsto \mathcal{A}$ is Σ^i_n– *computable* $(i = 0, 1, -1)$ if

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- *Rogers semilattice* $\mathcal{R}^i_n(\mathcal{A})$ of a family $\mathcal{A} \subseteq \Sigma^i_n$ is a quotient structure of all Σ^i_n–computable numberings of the family $\mathcal{A}$ modulo equivalence of the numberings ordered by the relation induced by reducibility of the numberings.
    - $\mathcal{R}^i_n(\mathcal{A})$ allows one to measure the different computations of a given family $\mathcal{A}$.
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- Usually, investigations in the theory of numberings use the following approach: given a family of sets $\mathcal{S}$ (say, Σ^0_n-computable and possessing some specific properties), they study various elementary and/or algebraic properties of the Rogers semilattice of all Σ^0_n-computable numberings of this particular $\mathcal{S}$.
- The main focus of our presentation contrasts with this approach: For a given level of complexity (say, Σ^0_α), we investigate the elementary theory of the semilattice $\mathcal{R}_{\Sigma^0_\alpha}$ that contains precisely all Σ^0_α-computable numberings of all Σ^0_α-computable families .

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we establish the complexity of the following first-order theories:

a) The theory $Th(\mathcal{R}_{\Sigma^0_1})$, where $\mathcal{R}_{\Sigma^0_1}$ is the semilattice of all computable numberings, is computably isomorphic to first order arithmetic .

b) The theory $Th(\mathcal{R})$, where $\mathcal{R}$ is the semilattice of all numberings, is computably isomorphic to second order arithmetic.

c) The theory $Th(\mathcal{SE})$, where $\mathcal{SE}$ is the commutative monoid of all computably enumerable equivalence relations (ceers) on $\mathbb{N}$, under composition, is computably isomorphic to first order arithmetic .

- For a structure $\mathcal{M}$, $Th(\mathcal{M})$ denotes the first order theory of $\mathcal{M}$. Recall that *first order arithmetic* is the theory $Th(\mathbb{N}; +, \times)$. It is known that first order arithmetic is m-equivalent to the set $\emptyset^{(\omega)}$ (i.e., the ω-jump of the empty set).
- For a set $X \subseteq \mathbb{N}$, let $\mathbf{R}_m^X$ denote the upper semilattice of X-c.e. m-degrees. Let $\mathbf{R}_m = \mathbf{R}_m^\emptyset$ (i.e., $\mathbf{R}_m$ is the semilattice of c.e. m-degrees). By $\mathbf{R}_m^X(\leq)$ we denote the partial order of X-c.e. m-degrees (in the language $\{\leq\}$).

Theorem (Nies ,1994)

The theory $Th(\mathbf{R}_m)$ is m-equivalent to first-order arithmetic.

For a computable language L, we use the following notations: K_L is the class of all L-structures, Sen_L is the set of all L-sentences, and Val_L is the set of all valid L-sentences. If n is a non-zero natural number, $\mathrm{C} \in \{\Sigma_n, \Pi_n\}$, and $\Gamma \subseteq Sen_L$, then

$$\mathrm{C}\text{–}\Gamma = \{\psi \in \Gamma : \psi \text{ is a } \mathrm{C}\text{-sentence}\}.$$

Second order arithmetic is the theory $Th(\mathcal{N}_2)$, where $\mathcal{N}_2 = (\mathbb{N} \cup P(\mathbb{N}); \mathbb{N}, P(\mathbb{N}), +, \times, \in)$. As usual, when working with $\mathcal{N}_2$, we treat it as a two-sorted structure. Variables $x, y, z, \ldots$ range over $\mathbb{N}$, and variables $X, Y, Z, \ldots$ range over $P(\mathbb{N})$.

Recall: A first order theory of finite signature is called hereditary undecidable if it is undecidable and any its subtheory of same signature is undecidable.

Let $\mathbf{D}_m$ denote the upper semilattice of all m-degrees.

Theorem (Nerode and Shore,1980)

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Theorem

The theory $Th(\mathcal{R}_{\Sigma^0_1})$ *is m-equivalent to first order arithmetic. Moreover, the fragment* Π_5*–*$Th(\mathcal{R}_{\Sigma^0_1})$ *is hereditarily undecidable.*

Lemma

There are binary relations $\trianglelefteq, \sim$*, and a binary function* $\tilde{\oplus}$ *with the following properties:*

1. $\sim$ *is an equivalence relation on* $\mathbb{N}$*;*
2. $\trianglelefteq$*,* $\sim$*, and* $\tilde{\oplus}$ *are arithmetical;*
3. *the quotient structure* $\mathcal{M} = (\mathbb{N}/\sim, \trianglelefteq, \tilde{\oplus})$ *is well-defined and isomorphic to the upper semilattice* $\mathcal{R}_{\Sigma^0_1}$*.*

Above lemma shows that the structure $\mathcal{R}_{\Sigma^0_1}$ has an arithmetical copy. This implies that $Th(\mathcal{R}_{\Sigma^0_1})$ is m-reducible to first order arithmetic.

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Lemma (follows from [?, Chapter 1, § 4])

Suppose that $A \neq B$ are c.e. sets. Then the semilattice $\mathcal{R}_{\Sigma^0_1}(\{A, B\})$ either has only one element, or is isomorphic to $\mathbf{R}_m$.

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The structure $\mathrm{R}_m(\leq)$ is Π_2-elementary definable with parameters in $\mathcal{R}_{\Sigma^0_1}$.

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Suppose that $\mathcal{A}$ is a subsemilattice of the structure $\mathcal{R} = (Num/\equiv, \leq, \oplus)$ such that $\mathcal{R}_{\Sigma^0_1}$ is a substructure of $\mathcal{A}$. Then the fragment Π_5–$Th(\mathcal{A})$ is hereditarily undecidable.

Corollary

Suppose that α is a computable ordinal such that $\alpha \geq 2$. Then the fragment Π_5–$Th(\mathcal{R}_{\Sigma^0_\alpha})$ is hereditarily undecidable.

Lemma (essentially follows from [?, Theorem 3.2])

Suppose that $A \neq B$ are Σ^0_α sets. Then the semilattice $\mathcal{R}_{\Sigma^0_\alpha}(\{A, B\})$ is isomorphic to one of the following two structures: either the semilattice of all Δ^0_α m-degrees, or $\mathbf{R}_m^{\emptyset^{(\alpha)}}$.

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The theory $Th(\mathcal{R})$ is 1-equivalent to second order arithmetic.

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The fragment Π_5–$Th(\mathcal{R})$ is hereditarily undecidable.

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The theory $Th(\mathcal{R})$ *is* 1*-equivalent to second order arithmetic.*

Corollary

The fragment $\Pi_5 - Th(\mathcal{R})$ *is hereditarily undecidable.*

We always consider equivalence relations with domain $\mathbb{N}$, if it is not specified otherwise. Let Id denote the identity relation on $\mathbb{N}$. A *ceer* is a computably enumerable equivalence relation.
If E and F are ceers, then the *composition* of E and F is the following binary relation:

$$E \circ F := \{(x, z) : \exists y[(xEy) \,\&\, (yFz)]\}.$$

It is easy to see that $E \circ F$ is also a ceer.
notation: Assume that $CEER$ is the set of all ceers. By $\mathcal{SE}$ we denote the structure $(CEER, \circ, Id)$.
It is not hard to prove that $\mathcal{SE}$ is a commutative monoid.

Let $\mathcal{EQ}$ denote the lattice of all ceers under inclusion:
$\mathcal{EQ} := (CEER, \subseteq, \cup, \cap)$.

Theorem (Carroll ,1986)

The theory $Th(\mathcal{EQ})$ is m-equivalent to first order arithmetic.

Nies (1994) proved that the upper semilattice of ceers modulo finite differences is also *m*-equivalent to first order arithmetic.

Lemma

The theory $Th(\mathcal{SE})$ *is m-equivalent to first order arithmetic.*

Lemma

The fragment $\Pi_5\text{–}Th(\mathcal{SE})$ *is hereditarily undecidable.*

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Thank you!