Decidable $\mathbf{0}'$-categoricity of models which realize only types with low CB ranks

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Decidable categoricity

Definition

A computable structure $\mathcal{A}$ is **d-computably categorical** (**d**-autostable) if for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists a d-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Goncharov investigated computable categoricity, restricted to decidable structures.

Definition

A decidable structure $\mathcal{A}$ is called **decidably d-categorical** (**d**-autostable relative to strong constructivizations) if every two decidable copies of $\mathcal{A}$ are d-computably isomorphic.

Categoricity spectrum

Definition(Fokina, Kalimullin, Miller, 2010)

The **categoricity spectrum** (autostability spectrum) of a computable structure $\mathfrak{M}$ is the set

$$\mathrm{CatSpec}(\mathfrak{M}) = \{\mathbf{d} : \mathfrak{M} \text{ is } \mathbf{d}\text{-computably categorical } \}.$$

A Turing degree $\mathbf{d_0}$ is the **degree of categoricity** of $\mathfrak{M}$ if $\mathbf{d_0}$ is the least degree in $\mathrm{CatSpec}(\mathfrak{M})$.

Spectrum of decidable categoricity

Definition(Goncharov, 2011)

The **decidable categoricity spectrum** (autostability spectrum relative to strong constructivizations) of the structure $\mathfrak{M}$ is the set

$$\mathrm{DecCatSpec}(\mathfrak{M}) = \{\mathbf{d} : \mathfrak{M} \text{ is decidably } \mathbf{d}\text{-categorical}\}.$$

A Turing degree $\mathbf{d_0}$ is the **degree of decidable categoricity** of $\mathfrak{M}$ if $\mathbf{d_0}$ is the least degree in $\mathrm{DecCatSpec}(\mathfrak{M})$.

Prime models and complete formulas

Let $\mathfrak{M}$ be a structure of a signature σ. $\mathrm{Th}(\mathfrak{M})$ denotes the first-order theory of $\mathfrak{M}$.

A structure $\mathfrak{M}$ is a **prime model** (of the theory $\mathrm{Th}(\mathfrak{M})$) if $\mathfrak{M}$ is elementary embeddable into every model $\mathfrak{N}$ of the theory $\mathrm{Th}(\mathfrak{M})$.

A structure $\mathfrak{M}$ is an **almost prime model** if there exists a finite tuple $\bar{c}$ from $\mathfrak{M}$ such that $(\mathfrak{M}, \bar{c})$ is a prime model.

A first-order formula $\psi(x_0, \ldots, x_n)$ is a **complete formula** for the theory $\mathrm{Th}(\mathfrak{M})$ if $\mathfrak{M} \models \exists \bar{x} \psi(\bar{x})$ and, for every σ-formula $\varphi(\bar{x})$, either $\mathfrak{M} \models \forall \bar{x}(\psi(\bar{x}) \to \varphi(\bar{x}))$ or $\mathfrak{M} \models \forall \bar{x}(\psi(\bar{x}) \to \neg\varphi(\bar{x}))$.

Nurtazin's criterion

Theorem (Nurtazin 1974)

Suppose that $\mathfrak{M}$ is a decidable structure of a signature σ. $\mathfrak{M}$ is decidably categorical if and only if there exists a finite tuple $\bar{c}$ from $\mathfrak{M}$ such that the following holds:

(a) The structure $(\mathfrak{M}, \bar{c})$ is a prime model of the theory $\mathrm{Th}(\mathfrak{M}, \bar{c})$.

(b) Given a $(\sigma \cup \{\bar{c}\})$-formula $\psi(\bar{x})$ one can effectively, uniformly in ψ, determine whether ψ is a complete formula for $\mathrm{Th}(\mathfrak{M}, \bar{c})$.

Goncharov's result

Theorem (Goncharov, 2011)

Let $\mathbf{d}$ be a Turing degree. Suppose that $\mathfrak{M}$ is a decidable structure of a language L, $\bar{a}$ is a finite tuple from $\mathfrak{M}$ such that the following conditions hold.

(a) The structure $(\mathfrak{M}, \bar{a})$ is a prime model.

(b) Given a $(L \cup \{\bar{a}\})$-formula $\psi(\bar{x})$, one can effectively relative to $\mathbf{d}$, uniformly in ψ, determine whether ψ is a complete formula in the theory $Th(\mathfrak{M}, \bar{a})$.

Then $\mathfrak{M}$ is decidably $\mathbf{d}$-categorical.

Known results

- Goncharov (2011) Every c.e. degree $\mathbf{d}$ is the degree of decidable categoricity of some decidable almost prime model of infinite signature.
- Bazhenov
  - For every computable ordinal α, the Turing degree $0^{(\alpha)}$ is a degree of decidable categoricity for some decidable Boolean algebra. (2016)
  - For a computable successor ordinal α, every Turing degree c.e. in and above $0^{(\alpha)}$ is the degree of decidable categoricity for some decidable structure. (2016)
  - For an infinite computable successor ordinal β, every Turing degree c.e. in and above $0^{(\beta)}$ is the degree of decidable categoricity for some linear order. (2017)
  - The set of all PA-degrees is the decidable categoricity spectrum. (2016).

Let T be a complete theory of a signature σ and p be the set of σ-formulas in free variables $x_1, \ldots, x_n$. We call p an (complete) **n-type** if the following holds:

(a) $p \cup T$ is satisfiable

(b) $\varphi \in p$ or $\neg\varphi \in p$ for all σ-formulas in free variables $x_1, \ldots, x_n$.

We let $\mathbf{S_n(T)}$ be the set of all n-types of T.

n-type p is said to be **principal** if it contains a complete formula.

Stone topology

Let T be a complete theory of a signature σ. For σ-formula in free variables $x_1, \ldots, x_n$, which is satisfiable with T, let

$$[\varphi] = \{p \in S_n(T) : \varphi \in p\}$$

The **Stone topology** (Mal'cev topology) on $S_n(T)$ is generating by taking the sets $[\varphi]$ as basic open sets.

Cantor-Bendixson rank

For a topological space X and an ordinal α, the **α-th Cantor-Bendixon derivative** of X is defined by transfinite induction as follows, where X' is the set of all limit points of X:

- $X^0 = X$
- $X^{\alpha+1} = (X^\alpha)'$
- $X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha$, for limit ordinals λ.

  The smallest ordinal α, such that $X^{\alpha+1} = X^\alpha$, is called the **Cantor-Bendixon rank** (CB-rank) of X.

Given a type $p \in S_n(T)$, we say that its **CB-rank** is α, written $CB(p) = \alpha$, if $p \in X^\alpha \setminus X^{\alpha+1}$.

Equivalently, $CB(p) = \alpha \Leftrightarrow p$ is an isolated point of $(S_n(T))^\alpha$. $CB(p) = 0 \Leftrightarrow p$ is a principal type of T.

Σ_2^0 Turing degree

Theorem 1

For every Σ_2^0 Turing degree $\mathbf{d}$, such that $\mathbf{d} \geq 0'$ there exists a decidable model $\mathfrak{M}$ such that:

(a) the set of complete formulas of $Th(\mathfrak{M})$ is computable,

(b) $\mathfrak{M}$ is not homogeneous and realizes 1-types only of Cantor-Bendixson rank ≤ 1

(c) $\mathbf{d}$ is a degree of decidable categoricity of $\mathfrak{M}$.

Theorem 2 (with N. Bazhenov)

Let $\mathfrak{M}$ be a model that realizes types only of Cantor-Bendixon rank ≤ 1 then $\mathfrak{M}$ is almost prime.

Thank you for your attention!