The strength of a theorem about subgraphs with nice properties

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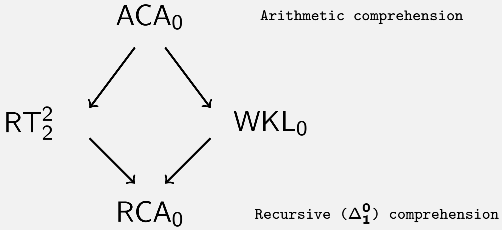
Overview

Rival-Sands' strength

RS for special graphs

RS-g and RT^2_2

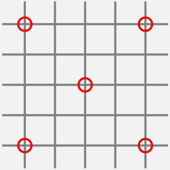
Which set existence axioms are *needed* to prove the theorems of ordinary mathematics?

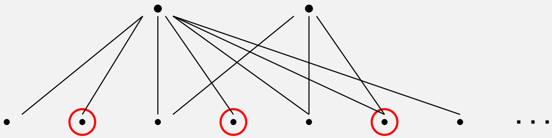


Rival-Sands for graphs

Theorem (RS-g)

For every infinite graph $G = (V, E)$, there is an infinite subgraph H such that every vertex in V is adjacent to at most one or to infinitely many vertices in H.





Rival-Sands' strength

Theorem

The following are equivalent:

1. ACA_0,
2. *for every infinite graph $G = (V, E)$, there is an infinite subgraph H such that every vertex in V is adjacent to at most one or to infinitely many vertices in H.*

Coding power

There exists a graph G such that each RS-g solution H codes $0'$.

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Questions

Understand what the coding power of RS-g is.

Give a Weihrauch-style analysis of RS-g.

Rival-Sands' strength: variants

Observation

The following are equivalent:

1. ACA_0,
2. *for every countable* ***locally finite*** *graph* $G = (V, E)$*, there is an infinite subgraph* H *such that every vertex in* V *is adjacent to at most one vertex in* H*.*

Observation (RCA_0)

For every infinite ***highly recursive*** *graph* $G = (V, E)$*, there is an infinite subgraph* H *such that every vertex in* V *is adjacent to at most one vertex in* H*.*

Rival Sands for comparability graphs

A poset of **width** n has no antichain of size $n+1$.

Theorem (RS-p)

*Let $(P, <_P)$ be a **poset** of finite width. Then there exists an infinite chain C such that every point in P is adjacent to none or to infinitely many points in C.*
Moreover, C may be chosen so that each $p \in P$ is comparable to none of the elements of C or to a cofinite subset of C.

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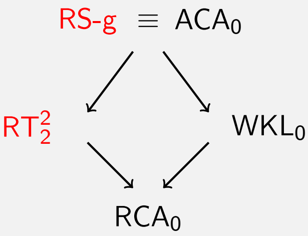
Question

What is the strength of RS-p?

Nice subgraphs

For every countable graph $G = (V, E)$,

- RT^2_2 *there is an infinite subgraph H such that H is complete or totally disconnected*
- RS-g *there is an infinite subgraph H such that every vertex in V is adjacent to at most one or to infinitely many vertices in H.*



Is it truly a Ramsey-type principle?

A statement of the form

$$\forall G(\varphi(G) \Rightarrow \exists H \psi(G, H))$$

is called **Ramsey-type** when it has the following properties.

- If $\varphi(G)$ and $\psi(G, H)$, then H must be infinite.
- If $\varphi(G)$, $\psi(G, H)$, and $H_0 \subseteq H$ is infinite, then also $\psi(G, H_0)$.

Is it truly a Ramsey-type principle?

Let G be an infinite graph and H be a subgraphs which is a solution for

RT^2_2 *then all subgraphs $H' \subseteq H$ are still solutions.*

RS-g *then not all subgraphs $H' \subseteq H$ are still solutions.*

But

- if G is **locally finite**, all subgraphs $H' \subseteq H$ are still solutions of RS-g.
- Solutions to RS-g are preserved under removing finitely many elements

I. Rival, B. Sands
On the adjacency of vertices to the vertices of an infinite subgraph.
J. London Math. Soc., 1980