Pathological Well-Orderings and Proof-Theoretic Ordinals

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- $PA \vdash TI(\prec, \Gamma)$ for $\prec$ representing R with $otyp(R) < \epsilon_0$

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For any r.e. theory $T \supseteq \mathrm{I}\Sigma_1$, we can find a representation $\prec$ of a well-ordered relation R with $otyp(R) = \omega$ such that $T \nvdash TI(\prec, \Delta^0_0)$.

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On the other hand, we will show, that for any $\alpha < \omega_1^{CK}$ it can be defined a representation $\prec$ of a well-ordered relation R in PA with $otyp(R) = \alpha$ such that $\mathrm{PA} \vdash TI(\prec, \Gamma)$.

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Let's sketch the proof!

Proof Sketch

Lemma (Beklemishev)

For any well-ordering $\langle \mathbb{N}, R \rangle$ *with* $otyp(R) < \omega_1^{CK}$, *there is a representation* $\prec$ *of* R *in* PA, *such that* $\mathsf{PA} \vdash \mathsf{Con}(I\Sigma_1 + TI(\prec, \Gamma))$.

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Let S *and* $T \supseteq \mathrm{PA}$ *be in* $L[\mathrm{PA}]$, *then there is a direct translation* f, *such that* $S \vdash \varphi \Rightarrow T + \mathrm{Con}(S) \vdash \varphi^f$.

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Theorem

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It happens, that $\mathrm{PA} \vdash \mathrm{Con}_{i\sigma_1 + ti(\psi,\Gamma)}$.

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$Th(S) \subseteq L$ and L is consistent.

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- Hcm_λ states, that the set represented by λ is Henkin-complete

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By Formalized Completeness, it holds that

$$\begin{aligned} &\mathsf{PA} + \mathsf{Con}_\sigma \vdash \varphi^f \leftrightarrow \lambda(\varphi) \\ S \vdash \varphi \Rightarrow\ &\mathsf{PA} + \mathsf{Con}_\sigma \vdash \lambda(\varphi) \\ \Rightarrow\ &\mathsf{PA} + \mathsf{Con}_\sigma \vdash \varphi^f \end{aligned}$$

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Are there no upper bounds for provable TI in Arithmetic?

Upper Bounds (Takeuti, Pohlers)

Let $\langle\mathbb{N}, R\rangle$ be a well-ordering and $\prec$ a recursive representation of R;

- $\bigcup\{otyp(R) \mid \mathrm{I}\Delta_0 \vdash TI(\prec,\Gamma)\} = \omega^2$
- $\bigcup\{otyp(R) \mid \mathrm{I}\Sigma_1 \vdash TI(\prec,\Gamma)\} = \omega^\omega$
- $\bigcup\{otyp(R) \mid \mathrm{PA} \vdash TI(\prec,\Gamma)\} = \epsilon_0$

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Not every recursive representation can be considered natural. And why should non-recursive representations be considered un-natural a priori?

Thank you!