Generic large cardinals as axioms

Monroe Eskew

Kurt Gödel Research Center
University of Vienna

July 24, 2018

Historical background

Theorem (Gödel, Cohen)

CH is independent of ZFC.

Historical background

Theorem (Gödel, Cohen)

CH is independent of ZFC.

Conjecture (Gödel)

Large cardinal axioms will settle CH.

Historical background

Theorem (Gödel, Cohen)

CH is independent of ZFC.

Conjecture (Gödel)

Large cardinal axioms will settle CH.

Theorem (Levy-Solovay)

CH is independent of ZFC + Large Cardinals.

Historical background

Theorem (Gödel, Cohen)

CH is independent of ZFC.

Conjecture (Gödel)

Large cardinal axioms will settle CH.

Theorem (Levy-Solovay)

CH is independent of ZFC + Large Cardinals.

By the work of Foreman, Woodin, and others, CH and many other low-level combinatorial statements are **not** independent of ZFC + "Generic Large Cardinals." Foreman proposed adopting these principles as axioms to settle classical independent questions.

The framework

Most sufficiently strong large cardinal hypotheses can be characterized in terms of an elementary embedding $j : N \to M$ between transitive classes with two parameters:

W: Where j sends ordinals.

Cl: How big N and M are.

The framework

Most sufficiently strong large cardinal hypotheses can be characterized in terms of an elementary embedding $j : N \to M$ between transitive classes with two parameters:

W: Where j sends ordinals.

CI: How big N and M are.

Generic large cardinal assumptions posit the elementary embedding to exist in a forcing extension. Thus there is a third parameter:

F: The nature (isomorphism type) of the forcing that introduces the embedding.

Foreman on GLC's

"The author contends that 'Generalized Large Cardinals' are straightforward generalizations of conventional large cardinals. Moreover, that the direct or indirect evidence for large cardinals, when suitably viewed, does not distinguish between conventional large cardinals and generic large cardinals and provides equally strong evidence for Generalized Large Cardinals."

–Matthew Foreman, "Has the continuum hypothesis been settled?" Logic Colloquium 2003.

Minimal generic hugeness

Definition (Foreman)

If μ is regular, $\kappa = \mu^+$ is called *minimally generically n-huge* if $\mathrm{Col}(\mu, \kappa)$ forces an embedding $j : V \to M \subseteq V[G]$ with critical point κ such that $M^{j^n(\kappa)} \cap V[G] \subseteq M$.

If $\kappa^{<\mu} = \kappa$, then $j^n(\kappa) = (\kappa^{+n})^V$.

Minimal generic hugeness

Definition (Foreman)

If μ is regular, $\kappa = \mu^+$ is called *minimally generically n-huge* if $\text{Col}(\mu, \kappa)$ forces an embedding $j : V \to M \subseteq V[G]$ with critical point κ such that $M^{j^n(\kappa)} \cap V[G] \subseteq M$.

If $\kappa^{<\mu} = \kappa$, then $j^n(\kappa) = (\kappa^{+n})^V$.

Theorem (Foreman)

If ω_1 is minimally generically 1-huge, then CH holds.

Minimal generic hugeness

Definition (Foreman)

If μ is regular, $\kappa = \mu^+$ is called *minimally generically n-huge* if $\mathrm{Col}(\mu, \kappa)$ forces an embedding $j : V \to M \subseteq V[G]$ with critical point κ such that $M^{j^n(\kappa)} \cap V[G] \subseteq M$.

If $\kappa^{<\mu} = \kappa$, then $j^n(\kappa) = (\kappa^{+n})^V$.

Theorem (Foreman)

If ω_1 is minimally generically 1-huge, then CH holds.

Similarly, we say κ is *minimally generically almost-huge* by replacing the closure property with "$M^{<j(\kappa)} \cap V[G] \subseteq M$."

Other generic hugeness, conflicts

Definition

Suppose μ is regular and $\kappa = \mu^+$.

1. κ is $\mathbb{P}$*-generically n-huge with target* λ if $\mathbb{P}$ forces an embedding $j : V \to M \subseteq V[G]$ with critical point κ such that $j(\kappa) = \lambda$ and $M^{j^n(\kappa)} \cap V[G] \subseteq M$.

Other generic hugeness, conflicts

Definition

Suppose μ is regular and $\kappa = \mu^+$.

1. κ is $\mathbb{P}$*-generically n-huge with target* λ if $\mathbb{P}$ forces an embedding $j : V \to M \subseteq V[G]$ with critical point κ such that $j(\kappa) = \lambda$ and $M^{j^n(\kappa)} \cap V[G] \subseteq M$.
   If n or λ is dropped, then by default we assume $n = 1$ and $\lambda = \kappa^+$.
   "Almost" can be used as before.

Other generic hugeness, conflicts

Definition

Suppose μ is regular and $\kappa = \mu^+$.

1. κ is $\mathbb{P}$*-generically n-huge with target* λ if $\mathbb{P}$ forces an embedding $j : V \to M \subseteq V[G]$ with critical point κ such that $j(\kappa) = \lambda$ and $M^{j^n(\kappa)} \cap V[G] \subseteq M$.
   If n or λ is dropped, then by default we assume $n = 1$ and $\lambda = \kappa^+$.
   "Almost" can be used as before.
2. κ is *canonically generically n-huge with target* λ if it satisfies the above with either $\mathbb{P} = \text{Col}(\mu, <\lambda)$ or $\text{Col}(\mu, \lambda^-)$.

Other generic hugeness, conflicts

Definition

Suppose μ is regular and $\kappa = \mu^+$.

1. κ is $\mathbb{P}$*-generically n-huge with target* λ if $\mathbb{P}$ forces an embedding $j : V \to M \subseteq V[G]$ with critical point κ such that $j(\kappa) = \lambda$ and $M^{j^n(\kappa)} \cap V[G] \subseteq M$.
   If n or λ is dropped, then by default we assume $n = 1$ and $\lambda = \kappa^+$.
   "Almost" can be used as before.
2. κ is *canonically generically n-huge with target* λ if it satisfies the above with either $\mathbb{P} = \mathrm{Col}(\mu, <\lambda)$ or $\mathrm{Col}(\mu, \lambda^-)$.

Theorem (Woodin)

If ω_1 *is minimally generically huge, then it is not* $\mathrm{Col}(\omega, <\lambda)$*-generically huge with target* λ *for any regular* λ*.*

More conflicts

Theorem (E.)

For any regular μ, if $\kappa = \mu^+$ is minimally generically almost-huge, then it is not $\mathbb{P}$-generically huge with target λ, for any regular λ and any $\mathbb{P}$ which is uniformly λ-dense and sufficiently absolutely λ-c.c. If λ is weakly inaccessible, then $\mathbb{P}$ cannot even be λ^+-c.c.

More conflicts

Theorem (E.)

For any regular μ, if $\kappa = \mu^+$ is minimally generically almost-huge, then it is not $\mathbb{P}$-generically huge with target λ, for any regular λ and any $\mathbb{P}$ which is uniformly λ-dense and sufficiently absolutely λ-c.c. If λ is weakly inaccessible, then $\mathbb{P}$ cannot even be λ^+-c.c.

Theorem (Woodin)

If ω_1 is minimally generically 3-huge, then ω_3 is not minimally-generically 1-huge.

More conflicts

Theorem (E.)

For any regular μ, if $\kappa = \mu^+$ is minimally generically almost-huge, then it is not $\mathbb{P}$-generically huge with target λ, for any regular λ and any $\mathbb{P}$ which is uniformly λ-dense and sufficiently absolutely λ-c.c. If λ is weakly inaccessible, then $\mathbb{P}$ cannot even be λ^+-c.c.

Theorem (Woodin)

If ω_1 is minimally generically 3-huge, then ω_3 is not minimally-generically 1-huge.

Theorem (E.)

For any regular μ, if $\kappa = \mu^+$ is canonically generically n-huge for $n \geq 2$, then for $0 < m < n$, κ^{+m} is not canonically generically almost-huge.

Justifications

How can these axioms be justified if they are mutually contradictory?

Justifications

How can these axioms be justified if they are mutually contradictory? Foreman again (1998):

"The advantage of generic large cardinals is that the critical point of j can be a 'small' cardinal such as $\aleph_1$. With some limitations this allows these cardinals to have similar reflection and resemblance properties as posited by large cardinal axioms on highly inaccessible cardinals. Moreover it allows one to state 'symmetry principles' that can hold in a generic extension of the universe."

Justifications

How can these axioms be justified if they are mutually contradictory? Foreman again (1998):

"The advantage of generic large cardinals is that the critical point of j can be a 'small' cardinal such as $\aleph_1$. With some limitations this allows these cardinals to have similar reflection and resemblance properties as posited by large cardinal axioms on highly inaccessible cardinals. Moreover it allows one to state 'symmetry principles' that can hold in a generic extension of the universe."

However:

1. The Π_1 statement that κ is a cardinal fails to reflect on a final segment of $\alpha < \kappa$ if κ is a successor cardinal.

Justifications

How can these axioms be justified if they are mutually contradictory? Foreman again (1998):

"The advantage of generic large cardinals is that the critical point of j can be a 'small' cardinal such as $\aleph_1$. With some limitations this allows these cardinals to have similar reflection and resemblance properties as posited by large cardinal axioms on highly inaccessible cardinals. Moreover it allows one to state 'symmetry principles' that can hold in a generic extension of the universe."

However:

1. The Π_1 statement that κ is a cardinal fails to reflect on a final segment of $\alpha < \kappa$ if κ is a successor cardinal.
2. There is a very simple sense in which the $\aleph_n$ for finite n do *not* resemble one another; each has a relatively simple definition.

Justifications

How can these axioms be justified if they are mutually contradictory? Foreman again (1998):

"The advantage of generic large cardinals is that the critical point of j can be a 'small' cardinal such as $\aleph_1$. With some limitations this allows these cardinals to have similar reflection and resemblance properties as posited by large cardinal axioms on highly inaccessible cardinals. Moreover it allows one to state 'symmetry principles' that can hold in a generic extension of the universe."

However:

1. The Π_1 statement that κ is a cardinal fails to reflect on a final segment of $\alpha < \kappa$ if κ is a successor cardinal.
2. There is a very simple sense in which the $\aleph_n$ for finite n do *not* resemble one another; each has a relatively simple definition.
3. "Symmetry" only appears by changing the background universe.

Multiversism

Recall the framework. A generic large cardinals is characterized by a generic embedding $j : M \to N$ with certain properties. Perhaps it is in the spirit of this idea to allow more variation of the *domain* of j.

Multiversism

Recall the framework. A generic large cardinals is characterized by a generic embedding $j : M \to N$ with certain properties. Perhaps it is in the spirit of this idea to allow more variation of the *domain* of j.

For example, we could assert that there is a universe V_1 in which ω_1 is minimally generically 2-huge, and another V_2 in which ω_2 is minimally generically 1-huge.

Multiversism

Recall the framework. A generic large cardinals is characterized by a generic embedding $j : M \to N$ with certain properties. Perhaps it is in the spirit of this idea to allow more variation of the *domain* of j.

For example, we could assert that there is a universe V_1 in which ω_1 is minimally generically 2-huge, and another V_2 in which ω_2 is minimally generically 1-huge.

This allows more methods for tackling consistency problems. Usually we start from Con(ZFC + LC), but we could start also from Con(ZFC + GLC). This has already been put into practice by Foreman and Woodin.

Thank you!