Categoricity Spectra for Linear Orders

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Categoricity spectra

Let $\mathbf{d}$ be a Turing degree. A computable structure $\mathcal{S}$ is $\mathbf{d}$-**computably categorical** if for any computable copy $\mathcal{A}$ of $\mathcal{S}$, there is a $\mathbf{d}$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{S}$.

The **categoricity spectrum** of $\mathcal{S}$ is the set

$$CatSpec(\mathcal{S}) = \{\mathbf{d} : \mathcal{S} \text{ is } \mathbf{d}\text{-computably categorical}\}.$$

A degree $\mathbf{c}$ is the **degree of categoricity** for $\mathcal{S}$ if $\mathbf{c}$ is the least degree in the spectrum $CatSpec(\mathcal{S})$.

Problem 1

Suppose that K is a familiar class of structures (e.g., abelian groups, distributive lattices, Boolean algebras, etc.). What categoricity spectra can be realized by structures from the class K?

Universal classes

We say that a class of structures K is **universal with respect to categoricity spectra** if for any computable structure $\mathcal{S}$, there is a computable structure $\mathcal{A}_{\mathcal{S}} \in K$ with

$$CatSpec(\mathcal{A}_{\mathcal{S}}) = CatSpec(\mathcal{S}).$$

Many familiar classes are universal with respect to categoricity spectra:

- directed graphs, symmetric irreflexive graphs, partial orders, (non-distributive) lattices, integral domains, commutative semigroups, 2-step nilpotent groups [Hirschfeldt, Khoussainov, Shore, Slinko 2002];
- fields of arbitrary characteristic [R. Miller, Poonen, Schoutens, Shlapentokh 2018];
- projective planes [Kogabaev 2015];
- structures with two equivalences [Tussupov 2016];
- polymodal algebras [B. 2016];
- ...

Non-universal classes

Some of familiar classes are non-universal with respect to categoricity spectra:

- Any computable equivalence structure has degree of categoricity $\mathbf{d} \in \{\mathbf{0}, \mathbf{0}', \mathbf{0}''\}$ [Csima, Ng].
- Every Δ^0_2-categorical Boolean algebra has degree of categoricity $\mathbf{d} \in \{\mathbf{0}, \mathbf{0}'\}$ [B. 2014].
- Any computable abelian p-group of a finite Ulm type has degree of categoricity $\mathbf{d} \in \{\mathbf{0}^{(n)} : n \in \omega\}$ [B., Goncharov, Melnikov].

Problem 1.a

What categoricity spectra can be realized by linear orders?

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Plan of the talk:

(a) Known degrees of categoricity for linear orders.
(b) Linear orders with no degree of categoricity.
(c) Non-strong degrees of categoricity.

Degrees of categoricity

Theorem (Fokina, Kalimullin, R. Miller 2010; Csima, Franklin, Shore 2013)

Let α be a computable ordinal.

(1) If α is non-limit and $\mathbf{d}$ is a Turing degree d.c.e. in and above $\mathbf{0}^{(\alpha)}$, then $\mathbf{d}$ is a degree of categoricity.

(2) $\mathbf{0}^{(\alpha)}$ is a degree of categoricity.

Theorem (Frolov)

Suppose that $2 \leq n < \omega$. If a degree $\mathbf{d}$ is d.c.e. in and above $\mathbf{0}^{(n)}$, then there is a computable linear order with degree of categoricity $\mathbf{d}$.

Degrees of categoricity

Theorem 1

Suppose that α is a computable successor ordinal with $\alpha > \omega$. If $\mathbf{d}$ is d.c.e. in and above $\mathbf{0}^{(\alpha)}$, then there is a computable linear order having degree of categoricity $\mathbf{d}$.

Note that the proof of Theorem 1 can be modified to obtain the Frolov's result for all finite $\alpha \geq 3$.

Corollary 1

Any degree $\mathbf{d}$ from Theorem 1 can be realized as degree of categoricity for an ordered abelian group.

Question 1

Suppose that $n \in \{0, 1\}$. Can every degree d.c.e. in and above $\mathbf{0}^{(n)}$ be realized as a degree of categoricity for a linear order?

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Note that recently, the following results were obtained:

(a) Any Δ_2^0 degree is a degree of categoricity [Csima, Ng].

(b) If δ is a limit ordinal and $\mathbf{d}$ is a degree *c.e.* in and above $\mathbf{0}^{(\delta)}$, then $\mathbf{d}$ is a degree of categoricity [Csima, Deveau, Harrison-Trainor, Mahmoud].

Structures with no degrees of categoricity

The first example of a computable structure with no degree of categoricity was constructed by R. Miller (2009).

Recall that a Turing degree $\mathbf{d}$ is a *PA-degree* if $\mathbf{d}$ computes a complete consistent extension of Peano arithmetic.

Theorem (R. Miller, Shlapentokh 2015)

There is a computable algebraic field such that its categoricity spectrum is equal to the set of PA-degrees.

Structures with no degrees of categoricity

Suppose that $X \subseteq \omega$. A degree $\mathbf{d}$ is a *PA-degree over* X if there is a $\mathbf{d}$-computable set A such that

$$\{e : \varphi_e^X(e)\downarrow = 1\} \subseteq A \quad \text{and} \quad \{e : \varphi_e^X(e)\downarrow = 0\} \subseteq \overline{A}.$$

Theorem (B. 2017)

Suppose that α is a computable successor ordinal such that $\alpha \geq 2$. There exists a computable distributive lattice such that its categoricity spectrum is equal to the set of PA-degrees over $\mathbf{0}^{(\alpha)}$.

Linear orders with no degree of categoricity

Example

Csima, Franklin, and Shore (2013) proved that any degree of categoricity is hyperarithmetical. It is known [Ash 1986] that the Harrison linear order $\mathcal{H} = \omega_1^{CK} \cdot (1 + \eta)$ has two computable copies which are not hyperarithmetically isomorphic. Therefore, the structure $\mathcal{H}$ does not have degree of categoricity.

Theorem 2

Suppose that α is a computable successor ordinal such that $\alpha \geq 4$. There exists a computable linear order such that its categoricity spectrum is equal to the set of PA-degrees over $\mathbf{0}^{(\alpha)}$.

Question 2

Suppose that $n \leq 3$. Can a categoricity spectrum of a linear order contain precisely the PA-degrees over $\mathbf{0}^{(n)}$?

Non-strong degrees of categoricity

Suppose that $\mathbf{d}$ is a Turing degree, and $\mathcal{S}$ is a computable structure. The degree $\mathbf{d}$ is the **strong degree of categoricity** for the structure $\mathcal{S}$ if:

1. $\mathbf{d}$ is the degree of categoricity for $\mathcal{S}$, and
2. there are two computable copies $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$ such that any isomorphism $f\colon \mathcal{A} \cong \mathcal{B}$ computes $\mathbf{d}$.

We say that $\mathbf{d}$ is the **non-strong degree of categoricity** for $\mathcal{S}$ if $\mathbf{d}$ is the degree of categoricity for $\mathcal{S}$, but not in a strong way.

Non-strong degrees of categoricity

The first examples of structures with non-strong degrees of categoricity were independently constructed by B., Kalimullin, Yamaleev (2018) and Csima, Stephenson:

Theorem (B., Kalimullin, Yamaleev 2018)

There exists a computable rigid graph G such that $\mathbf{0}'$ is the non-strong degree of categoricity for G.

Theorem (Csima, Stephenson)

There exists a rigid structure with computable dimension 3 and non-strong degree of categoricity $\mathbf{d} \leq \mathbf{0}''$.

Linear orders with non-strong degree of categoricity

Theorem 3

Suppose that α is a computable successor ordinal such that $\alpha \geq 4$. There exists a computable linear order $\mathcal{L}$ having a non-strong degree of categoricity $\mathbf{d}$ with $\mathbf{0}^{(\alpha)} \leq \mathbf{d} \leq \mathbf{0}^{(\alpha+1)}$.

We conjecture that the result can be refined by obtaining $\mathbf{d} = \mathbf{0}^{(\alpha+1)}$.

Decidable structures

A computable structure $\mathcal{S}$ is *decidable* if given a first-order formula $\psi(\bar{x})$ and a tuple $\bar{a}$ from $\mathcal{S}$, one can effectively determine whether $\psi(\bar{a})$ is true in $\mathcal{S}$.

Let $\mathbf{d}$ be a Turing degree. A decidable structure $\mathcal{A}$ is **decidably $\mathbf{d}$-categorical** if for any decidable copy $\mathcal{B}$ of $\mathcal{A}$, there is a $\mathbf{d}$-computable isomorphism $f\colon \mathcal{A} \cong \mathcal{B}$. The **decidable categoricity spectrum** of $\mathcal{A}$ is the set

$$DecCatSpec(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is decidably } \mathbf{d}\text{-categorical}\}.$$

A degree $\mathbf{c}$ is the **degree of decidable categoricity** for $\mathcal{A}$ if $\mathbf{c}$ is the least degree in the set $DecCatSpec(\mathcal{A})$.

Decidable categoricity spectra and degrees of decidable categoricity were introduced by Goncharov (2011).

Decidable categoricity for linear orders

A standard transformation $\mathcal{L} \mapsto \zeta \cdot \mathcal{L}$, where ζ is the ordering of integers, can be used to obtain some counterparts of Theorems 1-3 in the realm of decidable categoricity: for example,

Corollary 2

Let α be a computable successor ordinal with $\alpha > \omega$. If $\mathbf{d}$ is a Turing degree which is d.c.e. in and above $\mathbf{0}^{(\alpha)}$, then $\mathbf{d}$ is the degree of decidable categoricity for some discrete linear order.

Corollary 3

Let α be a computable successor ordinal with $\alpha > \omega$. There is a decidable linear order such that its decidable categoricity spectrum is equal to the set of PA-degrees over $\mathbf{0}^{(\alpha)}$.

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