Comparing the degrees of enumerability and the closed Medvedev degrees

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Reducibilities and degrees

Turing reducibility: For $f, g\colon \omega \to \omega$

$f \leq_{\mathrm{T}} g$ if there is a Turing functional Φ such that $f = \Phi(g)$.

Enumeration reducibility: For non-empty $A, B \subseteq \omega$

$A \leq_{\mathrm{e}} B$ if there is an enumeration operator Ψ such that $A = \Psi(B)$.

Equivalently: $A \leq_{\mathrm{e}} B$ if there is a Turing functional Φ such that $\operatorname{ran}\Phi(g) = A$ when $\operatorname{ran} g = B$.

Medvedev reducibility: For $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$

$\mathcal{A} \leq_{\mathrm{s}} \mathcal{B}$ if there is a Turing functional Φ such that $\Phi(g) \in \mathcal{A}$ when $g \in \mathcal{B}$.

For each $\leq_\circ$ above, define the corresponding degrees:

- $A \equiv_\circ B$ if $A \leq_\circ B$ and $B \leq_\circ A$.
- $\deg_\circ(A) = \{B : B \equiv_\circ A\}$.

Embedding $\mathcal{D}_\mathrm{T}$ into $\mathcal{D}_\mathrm{e}$

Let

- $\mathcal{D}_\mathrm{T}$ = the Turing degrees.
- $\mathcal{D}_\mathrm{e}$ = the enumeration degrees.
- $\mathcal{D}_\mathrm{s}$ = the Medvedev degrees.

In $\mathcal{D}_\mathrm{e}$, $\deg_\mathrm{e}(A \oplus \overline{A})$ plays the role of $\deg_\mathrm{T}(A)$:

$$A \oplus \overline{A} \leq_\mathrm{e} B \oplus \overline{B} \text{ if and only if } A \leq_\mathrm{T} B.$$

So $\deg_\mathrm{T}(A) \mapsto \deg_\mathrm{e}(A \oplus \overline{A})$ embeds $\mathcal{D}_\mathrm{T}$ into $\mathcal{D}_\mathrm{e}$.

Theorem (Cai, Ganchev, Lempp, J. Miller, M. Soskova)

The range of the embedding $\deg_\mathrm{T}(A) \mapsto \deg_\mathrm{e}(A \oplus \overline{A})$ *is definable in* $\mathcal{D}_\mathrm{e}$.

Embedding $\mathcal{D}_\mathrm{T}$ into $\mathcal{D}_\mathrm{s}$

Recall Medvedev reducibility: For $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$

$\mathcal{A} \leq_\mathrm{s} \mathcal{B}$ if there is a Turing functional Φ such that $\Phi(g) \in \mathcal{A}$ when $g \in \mathcal{B}$.

In $\mathcal{D}_\mathrm{s}$, $\deg_\mathrm{s}(\{f\})$ plays the role of $\deg_\mathrm{T}(f)$:

$$\{f\} \leq_\mathrm{s} \{g\} \text{ if and only if } f \leq_\mathrm{T} g.$$

So $\deg_\mathrm{T}(f) \mapsto \deg_\mathrm{s}(\{f\})$ embeds $\mathcal{D}_\mathrm{T}$ into $\mathcal{D}_\mathrm{s}$.

Theorem (Dyment, Medvedev)

The range of the embedding $\deg_\mathrm{T}(f) \mapsto \deg_\mathrm{s}(\{f\})$ *is definable in* $\mathcal{D}_\mathrm{s}$.

Observe: $\{f\}$ is a closed subset of ω^ω, so the range of the embedding is contained within the Medvedev degrees of closed sets.

Embedding $\mathcal{D}_{\mathrm{e}}$ into $\mathcal{D}_{\mathrm{s}}$

Recall again:

- $A \leq_{\mathrm{e}} B$ if $\exists\,\Phi$ such that $\operatorname{ran} g = B \Rightarrow \operatorname{ran}\Phi(g) = A$.
- $\mathcal{A} \leq_{\mathrm{s}} \mathcal{B}$ if $\exists\,\Phi$ such that $g \in \mathcal{B} \Rightarrow \Phi(g) \in \mathcal{A}$.

For non-empty $A \subseteq \omega$, define

$$\mathcal{E}_A = \{f : \operatorname{ran} f = A\}.$$

In $\mathcal{D}_{\mathrm{s}}$, $\deg_{\mathrm{s}}(\mathcal{E}_A)$ plays the role of $\deg_{\mathrm{e}}(A)$:

$$\mathcal{E}_A \leq_{\mathrm{s}} \mathcal{E}_B \text{ if and only if } A \leq_{\mathrm{e}} B.$$

So $\deg_{\mathrm{e}}(A) \mapsto \deg_{\mathrm{s}}(\mathcal{E}_A)$ embeds $\mathcal{D}_{\mathrm{e}}$ into $\mathcal{D}_{\mathrm{s}}$.

Question (Rogers)

Is the range of the embedding $\deg_{\mathrm{e}}(A) \mapsto \deg_{\mathrm{s}}(\mathcal{E}_A)$ definable in $\mathcal{D}_{\mathrm{s}}$?

Degrees of enumerability versus closed degrees

In the Medvedev degrees, we define the **degrees of enumerability** and the **closed degrees**.

Degrees of enumerability:
Call $\mathcal{E}_A = \{f : \operatorname{ran} f = A\}$ the **problem of enumerability** of A.
Call $\mathbf{E}_A = \deg_{\mathrm{s}}(\mathcal{E}_A)$ the corresponding **degree of enumerability**.

Closed degrees:
Call $\mathbf{c} \in \mathcal{D}_{\mathrm{s}}$ **closed** if $\mathbf{c} = \deg_{\mathrm{s}}(\mathcal{C})$ for a closed $\mathcal{C} \subseteq \omega^\omega$.

We investigate: Where are the degrees of enumerability?
Specifically: Where are the degrees of enumerability with respect to the closed degrees?

Recall that every Medvedev degree in the range of the embedding of $\mathcal{D}_{\mathrm{T}}$ into $\mathcal{D}_{\mathrm{s}}$ is a closed degree.

Degrees of enumerability that are also closed degrees

It is easy to see that degrees of enumerability can be closed degrees:

$$\mathcal{E}_{A \oplus \overline{A}} \equiv_{\mathrm{s}} \{A\} \quad \text{for every set } A \subseteq \omega.$$

Call the Medvedev degrees of the form $\deg_{\mathrm{s}}(\{A\})$ the **degrees of solvability**.

The degrees of solvability are the Medvedev degrees that correspond to Turing degrees.

Can a degree of enumerability be closed in a **non-trivial** way?

That is, are there closed degrees of enumerability that are not also degrees of solvability?

Non-trivially closed degrees of enumerability

There are degrees of enumerability that are closed in a non-trivial way.

In fact, there are closed degrees of enumerability that do not bound degrees of solvability (except $\mathbf{0}$).

Theorem

There is a closed (in fact compact) degree of enumerability $\mathbf{E}_A >_{\mathrm{s}} \mathbf{0}$ that does not bound a non-zero degree of solvability:

$$\neg\exists \mathbf{s}(\mathbf{0} <_{\mathrm{s}} \mathbf{s} \leq_{\mathrm{s}} \mathbf{E}_A \textit{ and } \mathbf{s} \textit{ is a degree of solvability}).$$

Uniformly e-pointed trees and compactness

Definition (Modification of a definition by Montalbán)

A **uniformly e-pointed tree (w.r.t. functions)** is a tree $T \subseteq \omega^{<\omega}$ s.t.

- T is finitely branching with **no leaves** and
- there is an enumeration operator Ψ such that $\Psi(\mathrm{graph}\, g) = T$ for all $g \in [T]$.

(That is, every path through the tree can enumerate the tree, uniformly.)

Lemma

(1) *For $A \subseteq \omega$, $\mathbf{E}_A$ is a compact degree of enumerability if and only if there is a uniformly e-pointed tree T such that $A \equiv_{\mathrm{e}} T$.*

(2) *There is a uniformly e-pointed tree T that is* **not** *r.e. and is such that there is no non-recursive B with $B \oplus \overline{B} \leq_{\mathrm{e}} T$.*

$\mathbf{E}_T$ is then the desired nontrivially compact degree of enumerability.

Digression on cototality

Definition (This idea has a long history. See Andrews *et al.*)

- A set $A \subseteq \omega$ is **cototal** if $A \leq_{\mathrm{e}} \overline{A}$.
- An enumeration degree $\mathbf{d}$ is **cototal** if $\mathbf{d} = \deg_{\mathrm{e}}(A)$ for a cototal A.

Facts:

- An enumeration degree is cototal iff it contains a uniformly e-pointed tree (see also McCarthy for a more thorough study of cototality and pointedness).
- From the previous slide, a degree of enumerability $\mathbf{E}_A$ is compact iff $A \equiv_{\mathrm{e}} T$ for a uniformly e-pointed tree T.
- Therefore a degree of enumerability $\mathbf{E}_A$ is compact iff A has cototal enumeration degree.

The rest of the mess

We have shown that it is possible for a Medvedev degree to be both a closed degree and a degree of enumerability in a non-trivial way.

Also:

- There are closed degrees $\mathbf{c} >_{\mathrm{s}} \mathbf{0}$ such that there are no degrees of enumerability $\mathbf{E}_A$ with $\mathbf{0} <_{\mathrm{s}} \mathbf{E}_A \leq_{\mathrm{s}} \mathbf{c}$.

  (The degree $\mathbf{c} = \deg_{\mathrm{s}}(\{0,1\}$-valued DNR functions$)$ is a good example. It is uncountable and meet-irreducible.)

- There are degrees of enumerability $\mathbf{E}_A >_{\mathrm{s}} \mathbf{0}$ such that there are no closed degrees $\mathbf{c}$ with $\mathbf{0} <_{\mathrm{s}} \mathbf{c} \leq_{\mathrm{s}} \mathbf{E}_A$.

  (To do this, build a non-r.e. set A such that whenever $T \leq_{\mathrm{e}} A$ is a tree with no leaves, T must have an r.e. subtree with no leaves.)

Grazie Mille!

Thank you for coming to my talk!
Do you have a question about it?