Trends in the history of infinity

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Cantor $(Ord, +, \cdot, <)$

$$2\omega \neq \omega 2, \quad 1 + \omega \neq \omega + 1$$

$$\neg(1 < 2 \Rightarrow 1 + \omega < 2 + \omega)$$

$$\neg(1 < 2 \Rightarrow \omega < 2\omega)$$

Cantor's theorem. Normal sum, normal product. $(Ord, \oplus, \odot, <)$

$$\alpha = \omega^{\eta_1} \cdot p_1 + \ldots + \omega^{\eta_h} \cdot p_h,$$

where $\eta_1 > \ldots > \eta_h$, $\eta_i \in Ord$, $h, p_i, q_i \in \mathbb{N}$

$$\alpha = \omega^{\eta_1} \cdot p_1 + \ldots + \omega^{\eta_h} \cdot p_h,$$

$$\beta = \omega^{\eta_1} \cdot q_1 + \ldots + \omega^{\eta_h} \cdot q_h$$

$$\begin{aligned} \alpha \oplus \beta &=_{df} \omega^{\eta_1} \cdot (p_1 + q_1) + \ldots + \omega^{\eta_h} \cdot (p_h + q_h) \\ \alpha \odot \beta &=_{df} \sum_{1 \le i, j \le h} \omega^{\eta_i \oplus \eta_j} \cdot p_i q_j \end{aligned}$$

$$\alpha \oplus \beta = \beta \oplus \alpha, \quad \alpha \odot \beta = \beta \odot \alpha$$

$$\alpha < \beta \Rightarrow \alpha \oplus \gamma < \beta \oplus \gamma, \quad \alpha < \beta \Rightarrow \alpha \odot \gamma < \beta \odot \gamma$$

Cantor vs Euler. What is finite?

Cantor $(\mathbb{N}, +, \cdot, <) \to (Ord, +, \cdot, <)$

$$(\mathbb{N}, +, \cdot, <) \quad vs \quad (\mathbb{R}, +, \cdot, 0, 1, <)$$

Euler $(\mathbb{R}, +, \cdot, 0, 1, <) \to$ non-Archimedean field $\mathbb{R}^*$

Greek magnitudes $\mathfrak{M} = (M, +, <)$. Euclid's *Elements*, book 5

- E1 $(\forall x, y)(\exists n \in \mathbb{N})(nx > y)$,
- E2 $(\forall x, y)(\exists z)(x < y \Rightarrow x + z = y)$,
- E3 $(\forall x, y, z)(x < y \Rightarrow x + z < y + z)$,
- E4 $(\forall x)(\forall n \in \mathbb{N})(\exists y)(x = ny)$,
- E5 $(\forall x, y, z)(\exists v)(x : y :: z : v)$.

Descartes, *La Géométrie*, 1637

$$(M, + ::, <) \to (\mathbb{F}, +, \cdot, 0, 1, <)$$

Ordered field 1. Real-closed field

Df1. A commutative field $(\mathbb{F}, +, \cdot, 0, 1)$ together with a total order $<$ forms an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ when field operations are compatible with the order, i.e.

1. $x < y \Rightarrow x + z < y + z$,
2. $x < y,\ 0 < z \Rightarrow xz < yz$.

Df2. An ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ is real-closed when it is closed under square root operation and every odd-degree polynomial $f \in \mathbb{F}[x]$ has a root in $\mathbb{F}$.

1. $(\forall x > 0)(\exists y \in \mathbb{F})(y^2 = x)$,
2. $(\forall a_1 \in \mathbb{F})...(\forall a_{2n} \in \mathbb{F})(\exists x \in \mathbb{F})(x^{2n+1} + \sum_{i=1}^{2n} a_i x^i = 0)$.

Th. When $(\mathbb{F}, +, \cdot, 0, 1, <)$ is a real-closed field, then $<$ is the only total order on $\mathbb{F}$ that is compatitble with the field operations.

Ordered field 2. Infinitesimals and infinitely large numbers

In any ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ one can define the following subsets of $\mathbb{F}$:

$$\begin{aligned} \Omega &= \{x \in \mathbb{F} : (\forall n \in \mathbb{N})(|x| < \tfrac{1}{n})\}, \\ \mathbb{L} &= \{x \in \mathbb{F} : (\exists n \in \mathbb{N})(|x| < n)\}, \\ \Psi &= \{x \in \mathbb{F} : (\forall n \in \mathbb{N})(|x| > n)\}. \end{aligned}$$

Elements of these sets are called infinitesimals, limited and infinitely large numbers, respectively.

$$(\forall x \neq 0)(x \in \Omega \Leftrightarrow \ x^{-1} \in \Psi)$$

Ordered field 3. Archimedean axiom

Here are some equivalent forms of the Archimedean axiom:

A1 $(\forall x, y \in \mathbb{F})(\exists n \in \mathbb{N})(0 < x < y \Rightarrow nx > y)$,

A4 $\Omega = \{0\}$.

The field $(\mathbb{F}, +, \cdot, 0, 1, <)$ is non-Archimedean iff $\Omega \neq \{0\}$.

Th. The field of real numbers is the *biggest* Archimedean field. It is also a real-closed field

Non-Archimedean ordered field in Euler's 1748 *Introductio in analysin infinitorum*

When N is an infnite number, then $N-1$, $\frac{N}{2}$, $\sqrt{N}$, etc. are infinite numbers too.

$$N \in \Psi \Rightarrow N-1, \frac{N}{2}, \sqrt{N} \in \Psi$$

J. Conway, *On numbers and Games* (1976, 2001).
The ordered *field* ONAG

$ONAG = \{a : a \text{ is a surreal number}\}$

1. $Ord \subset ONAG$.

   $$\omega \in ONAG$$

2. **Th.** $(ONAG, +, \cdot, 0, 1, <)$ is the *biggest* ordered field.

   $$\mathbb{R} \subset \mathbb{R}^* \subset ONAG$$
   $$\omega - 1,\ \omega - \tfrac{1}{2},\ \tfrac{\omega}{2}$$

3. $(ONAG, +, \cdot, 0, 1, <)$ is a real-closed field.

   $$\sqrt{\omega} \in ONAG$$

$$\sqrt{\text{THE END}}$$