Superstable expansions of $(\mathbf{Z}, +, 0)$

Quentin Lambotte
Françoise Point

Institut de Mathématique
Université de Mons

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History and motivation

Let R be a (strictly increasing) sequence of natural numbers. We let $\mathscr{Z}_R$ denote the structure $(\mathbf{Z}, +, -, 0, R)$ and $\mathscr{Z}_{<,R}$ denote $(\mathbf{Z}, +, -, 0, <, R)$. Let R be either (q^n), $(n!)$ or Fib.

$\mathscr{Z}_{<,R}$ is

1. model complete (Semenov)
2. decidable (Semenov)
3. has q.e. (Point) and is NIP

$\mathscr{Z}_R$ is

1. superstable
   - Poizat ; Sklinos and Palacin for (q^n)
   - Sklinos and Palacin for $(n!)$
   - Conant ; L. and Point for Fib.
2. decidable (L. and Point)
3. has q.e. (L. and Point)

Regular sequences

Definition

Call a (strictly increasing) sequence $R = (r_n)$ of natural numbers *regular* if

($\star$) $\lim_{n\to\infty} r_{n+1}/r_n = \theta \in \mathbf{R}^{>1} \cup \{\infty\}$ and if θ is algebraic, then R satisfies a linear recurrence relation whose characteristic polynomial is the minimal polynomial of θ.

Remark

The characteristic polynomial of the recurrence $r_{n+k} = a_{k-1} r_{n+k-1} + \cdots + a_0 r_n$ *is* $X^k - a_{k-1}X^{k-1} + \cdots + a_0$.

Expansion by a regular sequence

Theorem

Let R be a regular sequence. Then $\mathbf{Z}_R$ is superstable. Furthermore, if R' is a sum of regular sequences, then $\mathbf{Z}_{R'}$ is also superstable.

The proof uses the strategy of Palacin and Sklinos (also used by Conant).

Ingredients of the proof

The main tool used by Palacin and Sklinos is

Theorem (Casanovas and Ziegler)

Let R be a set of natural numbers. Assume that

1. *R_{ind} is superstable, where R_{ind} is the structure whose domain is R with, for every group-definable set $X \subset \mathbf{Z}^n$, a predicate for $X \cap R^n$;*
2. *R is small : for all $a, b \in \mathbf{N}$, the set $a + b\mathbf{N}$ cannot be covered by a set of the form $n_1 R + \cdots + n_k R$.*

Then $\mathscr{Z}_R$ is superstable.

Ingredients of the proof

The main tool used by Palacin and Sklinos is

Theorem (Casanovas and Ziegler)

Let R be a set of natural numbers. Assume that

1. R_{ind} *is superstable, where* R^0_{ind} *is the structure whose domain is* R *with predicates for sets of the form* $\{\bar{x} | a_1 x_1 + \cdots + a_n x_n = 0\} \cap R^n$ *;*
2. R *is small : for all* $a, b \in \mathbf{N}$*, the set* $a + b\mathbf{N}$ *cannot be covered by a set of the form* $n_1 R + \cdots + n_k R$*.*

Then $\mathscr{Z}_R$ *is superstable.*

Equations in R

Proposition

R^0_{ind} is definably interpreted in $\mathscr{N} = (\mathbf{N}, s, 0)$, where $s(n) = n + 1$. Furthermore, $\mathscr{N}$ is superstable.

Operators and equations

When R is regular, we are able to show that the trace of an equation $a_1x_1 + \cdots + a_nx_n = 0$ is determined by simpler objects.

Definition

1. A solution $\bar{c}$ of $a_1x_1 + \cdots + a_nx_n = k$, $k \in \mathbf{Z}$, is *non degenerate* if for all proper subset I of $\{1, \ldots, n\}$, $\sum_{i \in I} a_ix_i \neq 0$.
2. An *operator* on R is a function $\mathrm{A} : \mathbf{N} \to \mathbf{Z}$ of the form $\mathrm{A}(n) = a_1S^{k_1}(n) + \cdots + a_dS^{k_d}(n)$, where $\bar{a} \in \mathbf{Z}$, $\bar{k} \in \mathbf{Z}$, and $S^k(n) = r_{n+k}$.

Operators and equations

When R is regular, the non degenerate solutions of $a_1x_1 + \cdots + a_kx_k = 0$ are determined by finitely many operators of the form $\mathrm{A}_{\bar{a},\bar{k}}(n) = a_0S^{k_0}(n) + \cdots + a_nS^{k_n}(n)$.

Proposition (Mann Property)

For all $\bar{a}$, there exists $\bar{k}_1, \ldots, \bar{k}_m$ such that for all $\bar{\ell} \in \mathbf{N}$, $(r_{\ell_1}, \ldots, r_{\ell_k})$ is a non degenerate solution of $a_1x_1 + \cdots + a_kx_k = 0$ iff for some $i \in \{1, \ldots, m\}$, there exist n such that

1. $S^{k_{ij}}(n) = r_{\ell_j}$ *for all* $j \in \{1, \ldots, k\}$ *;*
2. $\mathrm{A}_{\bar{a},\bar{k}_i}(n) = 0$.

This shows that we can concentrate on the behaviour of operators in order to interpret an equation in $\mathscr{N}$.

Operators

When R is a regular sequence, operators on R are easy to understand.

Lemma

Let A *be an operator on* R*. Then then the set* $\{n \in \mathbf{N} | \mathrm{A}(n) = 0\}$ *is finite or cofinite.*

We may assume that $\mathrm{A}(n) = a_0 r_n + a_1 r_{n+1} + \cdots + a_d r_{n+d}$. In this case, $\theta_A = \lim_{n\to\infty} \mathrm{A}(n)/r_n$ always exists and either

1. $\theta_A = \infty$ (when $\lim_{n\to\infty} r_{n+1}/r_n = \infty$) ;
2. or $\theta_A = P(\theta)$, where $\theta = \lim_{n\to\infty} r_{n+1}/r_n$ and $P(X) = a_0 + a_1 X + \cdots + a_d X^d$.

Operators

Remark

In case θ is algebraic, the previous computations explain why we assumed that R satisfies a recurrence relation whose characteristic polynomial is the minimal polynomial P_θ of θ.

Indeed, multiplying P_θ by another polynomial acts as a shift of the sequence R. Thus, as $P(\theta) = 0$ implies $P = P_\theta Q$, we have that $\mathrm{A}(n) = 0$ for all sufficiently large n whenever $P(\theta) = 0$.

Smallness

Proposition

For all $a, b \in \mathbf{N}$ *a set of the form* $a + b\mathbf{N}$ *cannot be covered by finitely many sets of the form* $\{z + \mathrm{A}_1(n_1) + \cdots + \mathrm{A}_k(n_k) | \bar{n} \in \mathbf{N}\}$*, where* A_i *is an operator and* $z \in \mathbf{Z}$*.*

This is a consequence of a stronger result (proved using the Mann Property) :

Proposition

The set $\{z + \mathrm{A}_1(n_1) + \cdots + \mathrm{A}_k(n_k) | \bar{n} \in \mathbf{N}\}$ *is not piecewise syndetic : it does not contain arbitrarily large sequences of bounded gaps.*

Comparison with a result of G. Conant

Independently of our work, G. Conant also generalized the results of Palacin and Sklinos.

Theorem (G. Conant)

$\mathscr{Z}_R$ is superstable whenever R is a geometrically sparse sequence, that is there exists a sequence $(\lambda_n) \in \mathbf{R}^{>0}$ such that

1. $\sup_{n \in \mathbf{N}} |r_n - \lambda_n| < \infty$ *and*
2. *the set* $\{\lambda_n / \lambda_m | m \leq n\}$ *is closed and discrete.*

For example, $\mathscr{Z}_R$ is superstable where $R = (\lfloor \pi^n \rfloor)$.
The proof uses also the Theorem of Casanovas and Ziegler.

Comparison with a result of G. Conant

The set of regular sequences has a non trivial intersection with the set of geometrically sparse sequences :

1. ($\theta = \infty$) any such sequence is geometrically sparse ;
2. (θ is algebraic) $r_{n+2} = 5r_{n+1} + 7r_n$, with $r_1 = 1$ and $r_0 = 0$ (here, $\theta = (5 + \sqrt{53})/2$).
3. (θ is transcendental) Assume (r_n) is a regular sequence such that $\sup_{n \in \mathbf{N}} |r_n - \tau\theta^n| < \infty$ (that is, (r_n) is also geometrically sparse). Then, $(r_n + n)$ is not geometrically sparse.

Quantifier elimination

Using the relation between equations and operators on R, we are able to establish quantifier elimination in an expansion by definition of $(\mathbf{Z}, +, -, 0, 1, D_n, R, S, S^{-1})$.

Theorem

$(\mathbf{Z}, +, -, 0, 1, D_n, R, S, S^{-1})$ *has quantifier elimination after adding predicates for formulas of the form*

$$\exists \bar{x} \in R \bigwedge_{i \in I_1} D_{n_i}(\mathrm{A}_i(x_i) + k_i) \wedge \bigwedge_{i \in I_2} \sum_{j=1}^{m} \mathrm{A}_{ij}(x_j) = y_i .$$

Quantifier elimination follows from the following two properties.

Proposition

1. *T_R has algebraically prime models : for all substructure $\mathscr{A}$ of a given model $\mathscr{M}$, there exists a model $\bar{\mathscr{A}}$ such that any embedding $f : \mathscr{A} \to \mathscr{M}'$, $\mathscr{M}'$ a model, extends to an embedding $\bar{f} : \bar{\mathscr{A}} \to \mathscr{M}'$;*
2. *T_R is 1-e.c. : for all pair of models $\mathscr{M}_0 \subset \mathscr{M}$, any definable subset of $\mathscr{M}$, defined with parameters in $\mathscr{M}_0$, has a non empty intersection with $\mathscr{M}$.*

As a corollary of quantifier elimination, we have :

Theorem

T_R is superstable.

Thank you !