Compactness principles and cardinal invariants

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Let us denote by $\mathfrak{c}$ the class function which maps κ to 2^κ, for every cardinal κ. We call $\mathfrak{c}$ the **continuum function**.

The effect of compactness principles at successor cardinals on the continuum function $\mathfrak{c}$ has been studied recently:[1] Suppose $C(\kappa^{++})$ is some compactness principle holding at κ^{++} (such as the tree property or stationary reflection), does it limit in any way the value of $\mathfrak{c}(\kappa)$? Or more generally, does it limit the values of $\mathfrak{c}(\lambda)$ for cardinals λ in a neighbourhood of κ?

[1] See for instance [FHS18b, FHS18a, HS18, GK, Ung12] for more information.

The compactness principles considered can be various: usually one considers at κ^{++} the **tree property** (every κ^{++}-tree has a cofinal branch), **stationary reflection** (every stationary subset of κ^{++} of ordinals cofinality at most κ reflects), or the **negation of the approachability property**.

In general, any combinatorial property which is desirable or interesting can be considered – thus obtaining more information about the property and also about the forcings involved.

Results obtained so far suggest that $\mathfrak{c}$ can be anything we want (except for some trivial restrictions). This flexibility of $\mathfrak{c}$ with respect to compactness principles is analogous to the flexibility of $\mathfrak{c}$ with respect to large cardinals (since cardinals with compactness principles are former larger cardinals this is to be expected).

Seeing that compactness principles do not restrict $\mathfrak{c}$, it is natural to ask whether other functions such as $\mathfrak{b}, \mathfrak{d}, \mathfrak{u}$, etc., which correspond to cardinal invariants studied at ω, can be manipulated equally freely.[2]

We will mention some original results in this area.

[2] Cummings and Shelah (APAL, 1995) studied the global behaviour of $\mathfrak{c}$, $\mathfrak{d}$ and $\mathfrak{b}$ in ZFC (with no regard for large cardinals).

Let us denote by $\mathfrak{u}$ the function which to every cardinal κ assigns the least cardinal λ such that there is a family B of subsets of κ of size λ whose upwards closure is a uniform ultrafilter on κ. We call $\mathfrak{u}$ the **ultrafilter function**.

The behaviour of $\mathfrak{u}$ is quite mysterious: it is for instance open if $\mathfrak{u}(\aleph_1)$ can be smaller than $\mathfrak{c}(\aleph_1)$.

In their paper "Universal graphs at the successor of a singular cardinal" (JSL, 68, 2003), Džamonja and Shelah developed a method for obtaining a model where κ is regular (in fact supercompact) and

$$\kappa^+ \ll \mathfrak{c}(\kappa) \text{ and } \mathfrak{u}(\kappa) = \kappa^+.$$ [3]

The method of construction was later simplified by Brooke-Taylor, Fischer, Friedman and Montoya (APAL, 168, 2017, [BTFFM17]), and the value of $\mathfrak{u}(\kappa)$ was shown to consistently have any reasonable value below $\mathfrak{c}(\kappa)$.

Building on this technique, we have proved:

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[3] It was not the main purpose of the paper so the result is not stated explicitly in the paper.

Theorem (work in progress)

Suppose κ is a Laver-indestructible supercompact cardinal. Then the following hold:

(i) *If $\lambda > \kappa$ is weakly compact, then there is a generic extension where only the cardinals in the interval (κ^+, λ) are collapsed, $\mathfrak{u}(\kappa) = \kappa^+$, $\mathfrak{c}(\kappa) = \kappa^{++} = \lambda$ and many compactness principles hold at κ^{++}.*

(ii) *If $\lambda > \kappa$ is supercompact and $\mu > \lambda$ has cofinality greater than κ, then for any regular $\kappa < \kappa^* < \mu$ there is a generic extension where only the cardinals in the interval (κ^+, λ) are collapsed, $\mathfrak{u}(\kappa) = \kappa^*$, $\mathfrak{c}(\kappa) = \mu$ and many compactness principles hold at κ^{++}.*

Let us sketch the proof of (i).

Let $\kappa < \kappa^+ < \lambda < \lambda^+$ be as in [BTFFM17], with λ being weakly compact. Let $\mathbb{P}$ be an iteration of length λ^+ defined as in [BTFFM17].

We will “Mitchell-ise” the forcing by adding the collapsing component with the $\leq \kappa$-support.

Definition

Let $\mathbb{P}$ be as in [BTFFM17]. Then $\mathbb{P}^*$ is a forcing with conditions (p, q) such that:

- $p \in \mathbb{P}$,
- q is function with domain $\mathrm{dom}(q)$ of size at most κ, with

$$\mathrm{dom}(q) \subseteq \lambda \setminus \bigcup\{A_\alpha \mid \alpha < \lambda \text{ inaccessible}\}, \tag{1}$$

  where A_α is the half-open ordinal interval $[\alpha, \alpha^+)$.For every $\alpha \in \mathrm{dom}(q)$, $q(\alpha)$ is a $\mathbb{P}_\alpha$-name for a condition in $\mathrm{Add}(\kappa^+, 1)^{V[\mathbb{P}_\alpha]}$.

The ordering is the usual Mitchell ordering: $(p, q) \leq (p', q')$ iff $p \leq_{\mathbb{P}} p'$ and the domain of q extends the domain of q' and for all $\alpha \in \mathrm{dom}(q')$,

$$p|\alpha \Vdash_{\mathbb{P}_\alpha} q(\alpha) \leq q'(\alpha).$$

The desired model is $V[\mathbb{P}^*_\alpha \downarrow p]$ for some $p \in \mathbb{P}$ and α between λ and λ^+ of cofinality κ^+.

Since $\mathbb{P}^*_\alpha \downarrow p$ can be written as $\mathbb{P}_\alpha \downarrow p * \dot{R}$ for some κ^+-distributive $\dot{R}$, $\mathfrak{u}(\kappa) = \kappa^+$ follows as in [BTFFM17]. The restriction on the domain of q in (1) is used to get a suitable representation of the forcing $k(\mathbb{P}^*_\alpha \downarrow p)$ where k is a weakly compact embedding with critical point λ; this is used to argue for the compactness principles at λ using a version of a quotient analysis.

Open questions and problems

There is a tension between compactness at κ and large powerset of λ for some $\lambda < \kappa$ (κ used to be a large cardinal and therefore strong limit). If we wish just for a larger number of subsets, i.e. for $\mathfrak{c}(\lambda) > \kappa$, it seems this does not restrict the possibility of having compactness principles at κ. Perhaps by asking about more complex properties, compactness at κ might have a more profound effect. The presented method seems general enough to ensure many patterns of cardinal invariants discussed in [BTFFM17] for the supercompact κ, but many questions are open. Let us state just two:

- Is there (consistently) a successor cardinal κ with $\mathfrak{u}(\kappa) < \mathfrak{c}(\kappa)$?
- Can the above result be globalized (non-trivially) to deal with a segment of cardinals? And if not for $\mathfrak{u}$ (which seems hard), then at least for $\mathfrak{b}$ or $\mathfrak{d}$ (perhaps building on the results of Cummings and Shelah referred to earlier)?

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