Decidability in the Substructure Ordering of Finite Graphs

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Outline

1. Introduction to Graph Orders
2. Decidability in the Induced Graph Order: Literature
3. Results via Reduction from Subword Order

Introduction to Graph Orders

Graph Orders

- Graph order: a first order structure $(\mathcal{G}, \leq)$ where $\mathcal{G}$ is the set of all isomorphism types of simple, finite, undirected graphs.
- $\leq$ interpreted as a partial order.
- The substructure (aka induced subgraph), subgraph and minor relations are partial orders on $\mathcal{G}$.
- Focus on first order theory of induced subgraph order in this talk.

Induced Subgraph Order $(\mathcal{G}, \leq)$

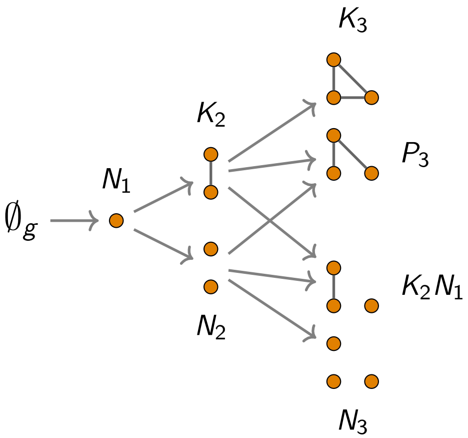


Figure: The first few levels of the induced subgraph order. Note the symmetry arising from the automorphism $f(g) = g^c$.

Induced Subgraph Order $(\mathcal{G}, \leq)$ as First Order Structure

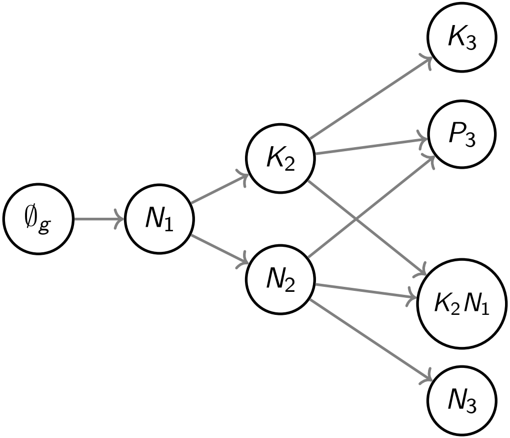


Figure: Initial layers of the structure $(\mathcal{G}, \leq)$. Note that the edge relation i.e. the "internal structure" is not available to us directly, nor are the names indicated inside the nodes. The arrows indicate the upper cover relation.

Characterization of Definability in $(\mathcal{G}, \leq, P_3)$

P_3: constant for path on three vertices, necessary to break the automorphism $f(g) = g^c$.

Theorem (T. 2017)

A relation $R \subseteq \mathcal{G}^n$ is definable in $(\mathcal{G}, \leq, P_3)$ if and only if it is arithmetical.

Corollary

Every graph is definable as a constant in $(\mathcal{G}, \leq, P_3)$.

Decidability in the Induced Graph Order: Literature

Undecidability of $(\mathcal{G}, \leq)$

Theorem (Wires 2016)

The first order theory of $(\mathcal{G}, \leq)$ is undecidable.

- The above result follows from definability of arithmetic in the induced subgraph order.
- The characterization theorem in T. 2017 does not follow from the above.

The Usual Questions

- Can we strengthen the undecidability result? What are the decidable fragments?
- Various possibilities:
  - Domain restrictions: $\mathcal{G}_0 \subset \mathcal{G}$, $(\mathcal{G}_0, \leq)$ decidable?
  - Vocabulary restrictions: $(\mathcal{G}, \lessdot)$ where $\lessdot$ is the covering relation.
  - Syntactic restrictions.
- Focus on syntactic fragments of $(\mathcal{G}, \leq)$: existential theory and finite variable fragments.

Results via Reduction from Subword Order

The Subword Order

- Let $\Sigma = \{a, b\}$ and Σ^* the set of all finite strings over Σ.
- For $w, w' \in \Sigma^*$, define $\leq_{sw}$ by

$$w \leq_{sw} w' \iff w \text{ can be obtained from } w' \text{ by deletion of an arbitrary number of letters.}$$

- $abab \leq_{sw} aabaab$ $\qquad$ $baba \not\leq_{sw} aabaab$
- $(\Sigma^*, \leq_{sw})$ is the subword order.

The Subword Order II

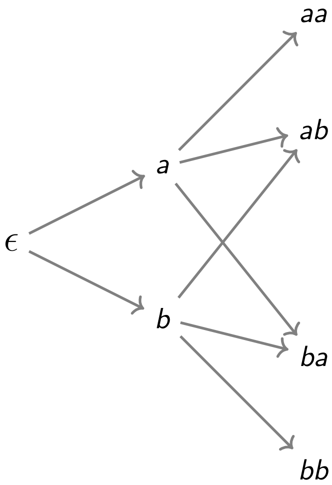


Figure: The first few levels of the subword order. Note the symmetry arising from the automorphism $f(w) = w_{a \leftrightarrow b}$ which swaps all a's and b's.

Words as Graphs

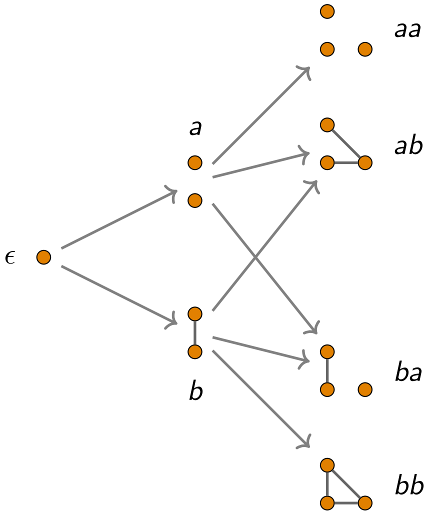


Figure: Concatenation by *a* as addition of isolated vertex, concatenation by *b* as addition of dominating vertex.

Threshold Graphs

Definition

A threshold graph is a non-empty graph which does not contain a ,  or  as induced subgraph. We denote the set of all threshold graphs by *Thr*.

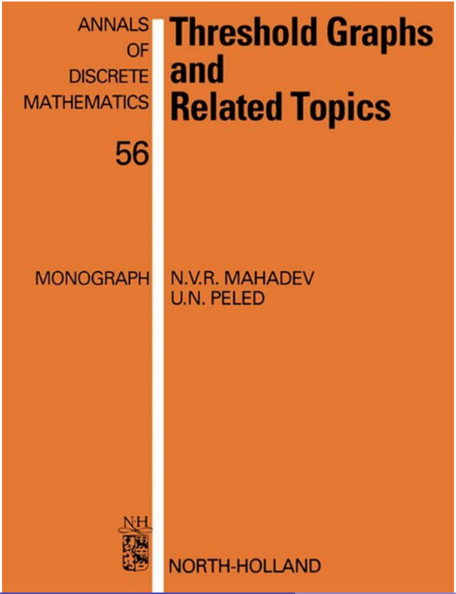
$$Thr(x) := x \neq \emptyset_g \ \wedge \ P_4 \not\leq x \ \wedge \ C_4 \not\leq x \ \wedge \ K_2K_2 \not\leq x$$

Lemma (Chvátal and Hammer 1975)

A graph g is a threshold graph if and only if it can be obtained from  by a finite sequence of operations of the kind:

1. AddIso: addition of an isolated vertex. AddIso() = 
2. AddDom: addition of a dominating vertex. AddDom() = 

Order Preserving Bijection between Threshold Graphs and Words



$$(Thr, \leq) \cong (\Sigma^*, \leq_{sw})$$

Quantifier-free interpretation of sub-word order in induced subgraph order.

Summary of Decidability Results for the Subword Order

Theorem (Halfon etal 2017)

Existential theory of $(\Sigma^*, \leq_{sw}, \mathscr{C}_w)$ is undecidable.

Proof: Existential interpretation of arithmetic.

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Theorem (Karandikar and Schnoebelen 2017)

FO^2 theory of $(\Sigma^*, \leq_{sw}, \mathscr{C}_w)$ is decidable.

Proof: Closure properties of piecewise-testable languages.

Our Results

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FO^2 theory of $(\mathcal{G}, \leq, \mathscr{C}_g)$.

?

Related to Ramsey theory.

Summary

- The full first order theory of $(\mathcal{G}, \leq)$ is extremely powerful and can be used to formalize many theorems in graph theory.
- Undecidability results can be obtained by reduction from the subword order.
- Decidability results will generalize those known for the subword order.
- The decidability of the FO^2 fragment of $(\mathcal{G}, \leq, \mathscr{C}_g)$ is open and looks very interesting.