Computable metrics above the standard real metric

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Contents

Computable categoricity of metric spaces

In Pour-El and Richards's *Computability in Analysis and Physics* (1989), the problem of computable categoricity of Banach spaces was approached in the following setting. Computable Banach space $\mathcal{B} = (B, \|\cdot\|, +, (r\cdot)_{r\in\mathbb{Q}})$ is called **computably categorical** if any two countable dense subsets of B, with respect to which space operations are computable, are computably isometric. Since then, a number of results on computable categoricity for Banach and metric spaces has been obtained within this setting.
We are motivated by the following

Question 1

What can be said about computable categoricity of a space when it's viewed as a completion of a canonical countable set by different metrics?

Computable categoricity

Countable computable model $\mathfrak{M}$ is **computably categorical** (or **autostable**) if any computable model $\mathfrak{N}$ isomorphic to $\mathfrak{M}$ is computably isomorphic to it.

The number of computable copies of $\mathfrak{M}$ up to computable isomorphism is called the **computable dimension** of $\mathfrak{M}$.

Theorem 1.1

$\langle \mathbb{Q}, \leqslant \rangle$ *is computably categorical.*

Theorem 1.2 (Fröhlich, Shepherdson)

There exists a computable field that is not computably categorical.

Theorem 1.3 (Maltsev)

There exists a computable abelian group that is not computably categorical.

Computable categoricity

Theorem 1.4 (Nurtazin)

A decidable structure either has computable dimension 1 or ω.

Theorem 1.5 (Nurtazin; Metakides, Nerode; Goncharov; Goncharov, Dzgoev; LaRoche; Remmel)

Structures of the following classes either have computable dimension 1 or ω: algebraically closed fields; real closed fields; abelian groups; linear orderings; Boolean algebras; Δ^0_2-categorical structures.

Theorem 1.6 (Goncharov)

For all $n > 1$, there exists a computable structure of computable dimension n.

Pour-El and Richards's approach

Computability in Analysis and Physics, 1989

Pour-El and Richards studied the question of uniqueness of "effectively separable computability structure" in computable Banach space $\mathcal{B}$ up to computable isometry.

Essentially, an effectively separable computability structure is a collection of all computable sequences in $\mathcal{B}$, generated by a countable basis of $\mathcal{B}$.

Effectively separable computability structures $\mathcal{S}_1$ and $\mathcal{S}_2$ is are **computably isometric** if there is an isometry $U : B \to B$ such that any sequence $(f_n)_{n\in\omega}$ is computable in $\mathcal{S}_1$ if and only if $f_n = U(g_n)$ for some $(g_n)_{n\in\omega}$ computable in $\mathcal{S}_2$.

Pour-El and Richards's approach

Theorem 1.7 (Pour-El, Richards)

- *All computability structures in computable Hilbert space are pairwise computably isometric.*
- *However, there exists a structure in the space l_1 that is not computably isometric to the standard structure of this space.*

Computable categoricity of metric spaces

Z. Iljazović, Isometries and Computability Structures, 2010

Theorem 1.8

All computability structures in an effectively compact computable metric space are pairwise computably isometric.

A. Melnikov, *Computably Isometric Spaces*, 2013

Metric space X is computably categorical if any two computability structures in it are computably isometric.

Theorem 1.9

- *Hilbert space is computably categorical as a metric space.*
- l_1 *is not computably categorical as a metric space.*
- $C[0,1]$ *is not computably categorical as a metric space.*

Computable categoricity of metric spaces

A. Melnikov, K. M. Ng, *Computable structures and operations on the space of continuous functions*, 2015

Theorem 1.10

- $C[0,1]$ *has computable dimension* ω.
- $C[0,1]$ *is not computably categorical as a Banach space.*
- $(C[0,1],+,\times,0,1)$ *is not computably categorical as a Banach algebra.*

T. McNicholl, *A note on the computable categoricity of* l_p *spaces*, 2015

Theorem 1.11

- l_p *is* Δ^0_2*-categorical for computable* p.
- l_p *is computably categorical iff* $p = 2$.

Representations

Definition 2.1

*A **computable functional** is a partial function* $\Phi\colon \omega^\omega \to \omega^\omega$ *such that for some oracle computable function* φ_e

$$\Phi(f) = g \text{ iff } \varphi_e^f(n) = g(n) \text{ for all } n.$$

Definition 2.2

*A **representation** of a set* X *is a partial surjection* $\delta\colon \omega^\omega \to X$.

Definition 2.3

A partial function $F\colon X \to Y$ *is* (δ_X, δ_Y)*-**computable** if there exists a computable functional* Φ *such that*

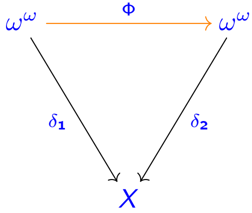
$$F\delta_X(f) = \delta_Y\Phi(f) \text{ for } f \in dom(F\delta_X).$$

Reducibility of representations

Definition 2.4

*Let δ_1, δ_2 be representations. δ_1 is **computably reducible** to δ_2 ($\delta_1 \leq_c \delta_2$) if there exists a computable functional Φ such that*

$$\delta_1(f) = \delta_2\Phi(f) \text{ for } f \in dom(\delta_1)$$



or, equivalently, if the identity function id_X is (δ_1, δ_2)-computable.

Definition 2.5

Let (X, ρ) be a complete separable metric space with a dense countable subset $W \subseteq X$, $W = (w_n)_{n\in\omega}$.

*The space $\mathbf{X} = (X, \rho, W)$ is called an **effective metric space**.*

*If the distance function $\rho(w_n, w_m) \in \mathbb{R}_c$ is computable in n and m, effective space $\mathbf{X}$ and metric ρ are called **computable**.*

Cauchy representations

Definition 2.6

Cauchy representation *$\delta_\rho\colon \omega^\omega \to X$ is defined as follows: for $x \in X$ and $f \in \omega^\omega$ we say that f is a* ***Cauchy name*** *for x, or $\delta_\rho(f) = x$, if*

$$w_{f(n)} \to x \text{ and } \rho(w_{f(n)}, w_{f(m)}) \le 2^{-n} \text{ for } m > n,$$

i.e. $w_{f(n)}$ quickly converges to x.

Let (X, ρ_1, W) and (X, ρ_2, W) be effective metric spaces. We say $\rho_1 \le_c \rho_2$ if $\delta_{\rho_1} \le_c \delta_{\rho_2}$.

Lemma 2.1

If $\exists M > 0\ \forall x, y \in X$

$$\rho_2(x, y) \leqslant M \cdot \rho_1(x, y)$$

(id_X is Lipschitz continuous w.r.t. δ_{ρ_2} and δ_{ρ_1}), then $\rho_1 \le_c \rho_2$.

Reducibility $\leq_{ch}$

Let $\delta\colon \omega^\omega \to X$ be a representation. The **final topology** of δ is the finest topology τ_δ of X with respect to which δ is continuous.

Definition 2.7

Let representations X δ_1 and δ_2 have the same final topology. We say that $\delta_1 \leq_{ch} \delta_2$ if there exists a (δ_1, δ_2)-computable autohomeomorphism of X.

Lemma 2.2

If $\delta_1 \leq_c \delta_2$, then $\delta_1 \leq_{ch} \delta_2$.

Proof.

$\delta_1 \leq_c \delta_2$ means that id_X is a (δ_1, δ_2)-computable homeomorphism. □

Metrics that admit no computable homeomorphism

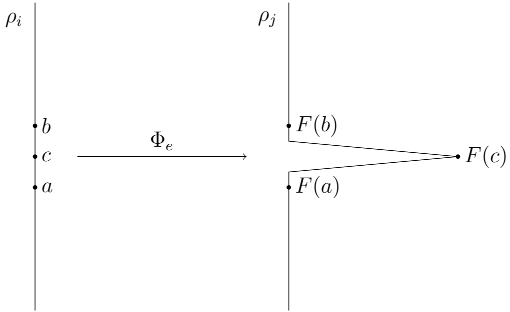
Theorem 2.1

There exists a countable anti-chain $(\rho_i)_{i \in \omega}$ of computable metrics, incomparable to each other w.r.t $\leq_{ch}$ and c-reducible to $\rho_{\mathbb{R}}$.

Real line (with rationals as a dense subset) has computable dimension ω.

Proof idea

We prevent Φ_e from $(\delta_{\rho_i}, \delta_{\rho_j})$-computing a real homeomorphism, for each e, i, j. Individual strategy for e, i, j runs on its own distinct interval in $\mathbb{R}$.



If we suspect that Φ_e computes a homeomorphism F on this interval, we change the approximation for ρ_j on it, corrupting a Cauchy name for the image of a certain element.

Metrics that admit no computable homeomorphism

Proceeding in this manner, we make sure that Φ_e cannot compute a real homeomorphism and thus violate the reducibility of ρ_i to ρ_j by Φ_e.

Finite priority method is used to eliminate possible conflicts of these strategies.

On the other hand, metrics ρ_i are constructed in a way that $\rho_{\mathbb{R}}(x, y) \leqslant \rho_i(x, y)$ for all x, y, so $\rho_i \leq_c \rho_{\mathbb{R}}$.

Computable metrics above $\rho_{\mathbb{R}}$

The previous result implies that $\rho_{\mathbb{R}}$ is not a minimal computable metric. We want to know whether we still can somehow simply characterize it in terms of representation reducibility. E.g. can we show that it is maximal or greatest?

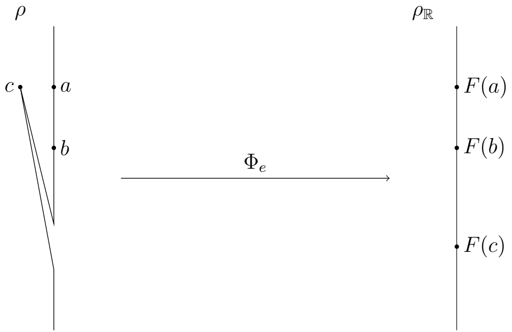
However, a construction very similar to the previous one shows that it is not true either.

Theorem 3.1

There exists a computable metric $\rho >_{ch} \rho_{\mathbb{R}}$.

Proof idea

Instead of spoiling Cauchy names by increasing the distances, we now do it by introducing new names that are absent in the standard metric.



Wait until we know that a and c are mapped to different locations in $\mathbb{R}$, then make them close to each other in ρ, contradicting the fact that Φ_e computes a homeomorphism.

Computable metrics above $\rho_{\mathbb{R}}$

Theorem 3.2

$\omega^{<\omega}$ is isomorphically embeddable into the ordering $\leq_{ch}$ of computable metrics above $\rho_{\mathbb{R}}$.

Hypothesis 1

Any finite partial ordering is isomorphically embeddable into the ordering $\leq_{ch}$ of computable metrics above $\rho_{\mathbb{R}}$.

Reducibility $\leq_c$

Lemma 3.1

Computable metrics form a lower semilattice under reducibility $\leq_c$*.*

Theorem 3.3

The class of c*-inequivalent computable metrics is effectively infinite (i.e. for any computable sequence* ρ_i *of computable metrics we can construct a metric* ρ *such that* $\rho \not\equiv_c \rho_i$ *for all* i*).*