Group Representation and Hahn-type Embedding for a class of Involutive Residuated Chains with an Application in Substructural Fuzzy Logic

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Substructural Logics

- Substructural logics encompass among many others, classical logic, intuitionistic logic, relevance logics, many-valued logics, mathematical fuzzy logics, linear logic along with their non-commutative versions.
- Algebraic counterpart:
  Residuated Lattices or FL-algebras

FL-algebras

An algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, t, f)$ is called a *full Lambek algebra* or an *FL-algebra*, if

- $(A, \wedge, \vee)$ is a lattice (i.e., $\wedge$, $\vee$ are commutative, associative and mutually absorptive),
- $(A, \cdot, t)$ is a monoid (i.e., $\cdot$ is associative, with unit element t),
- $x \cdot y \leq z$ iff $y \leq x \backslash z$ iff $x \leq z / y$, for all $x, y, z \in A$,
- f is an arbitrary element of A.

Residuated lattices are exactly the f-free reducts of FL-algebras. So, for an FL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, t, f)$, the algebra $\mathbf{A}_r = (A, \wedge, \vee, \cdot, \backslash, /, t)$ is a residuated lattice and f is an arbitrary element of A. The maps $\backslash$ and $/$ are called the *left* and *right division*.

- commutative: $x \rightarrow y$

Group-like FL_e-chains

- An FL_e-algebra is a commutative FL-algebra.
- An FL_e-chain is a totally ordered FL_e-algebra.
- An FL_e-algebra is called involutive if $x'' = x$ where $x' = x \to f$
  An FL_e-algebra is called group-like if it is involutive and
  $f = t$ (note that $f' = t$)

Group-like FL_e-chains

- Examples of FL_e-chains are f.o. groups or odd Sugihara chains,

18. J. M. Font, G. R. Prez, A note on Sugihara algebras, Publicacions Matemtiques, 36 vol. 2A (1992), 591–599

4. W. J. Blok and W. Dziobiak, On the Lattice of Quasivarieties of Sugihara Algebras, Studia Logica: An International Journal for Symbolic Logic, 45 vol. 3 (1986), 275–280

- distinguished by the number of idempotent elements

Relation of group-like FL_e-algebras to abelian groups

Theorem 1. *For a group-like FL_e-algebra $(X, \wedge, \vee, \circledast, \rightarrow_{\circledast}, t, f)$ the following statements are equivalent:*

1. *Each element of X has inverse given by $x^{-1} = x'$, and hence $(X, \wedge, \vee, \circledast, t)$ is a lattice-ordered Abelian group,*
2. *$\circledast$ is cancellative,*
3.
4. *The only idempotent element in the positive cone of X is t.*

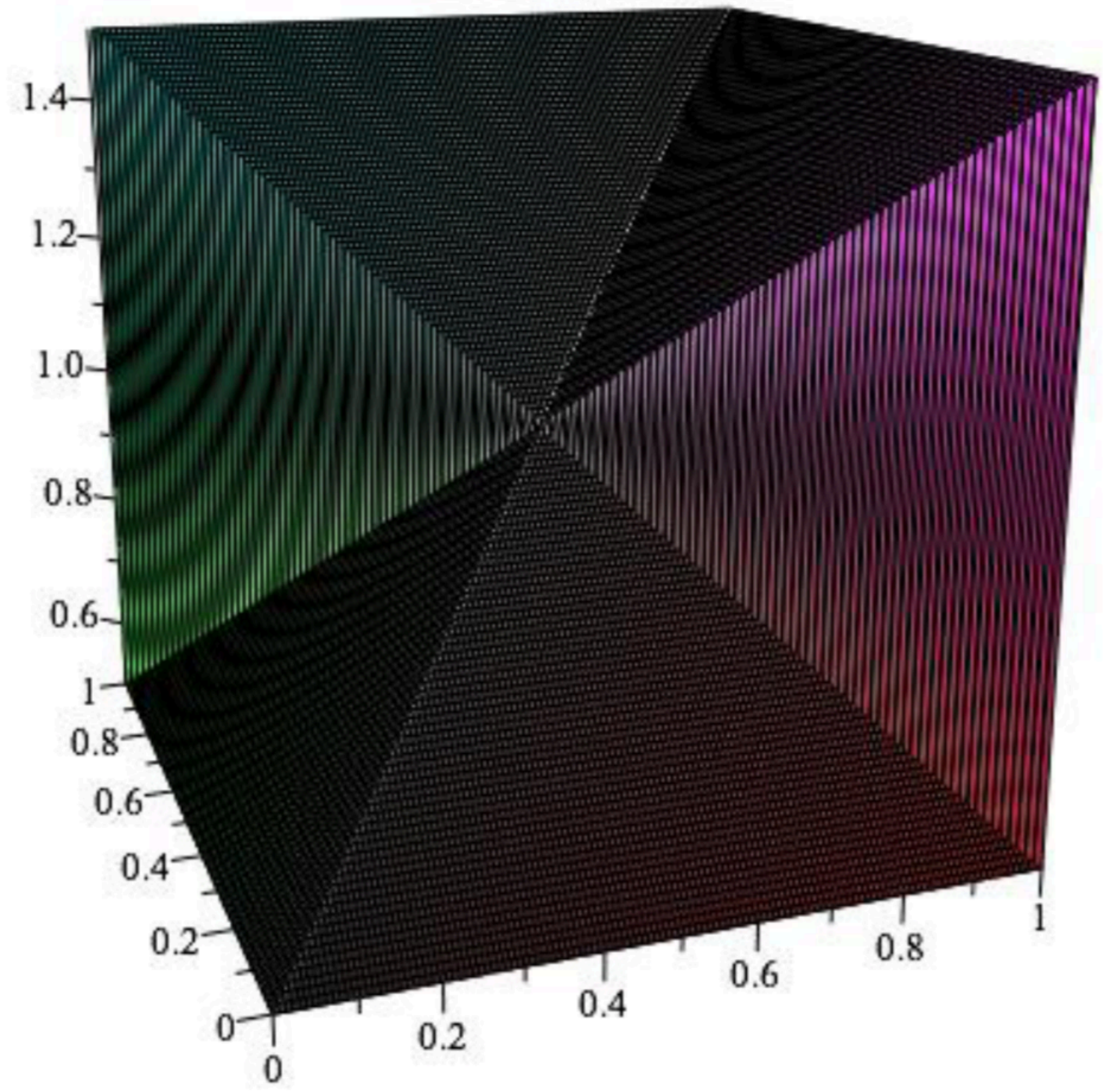


Fig. Visualization: The only odd Sugihara algebra over $]0,1[$.

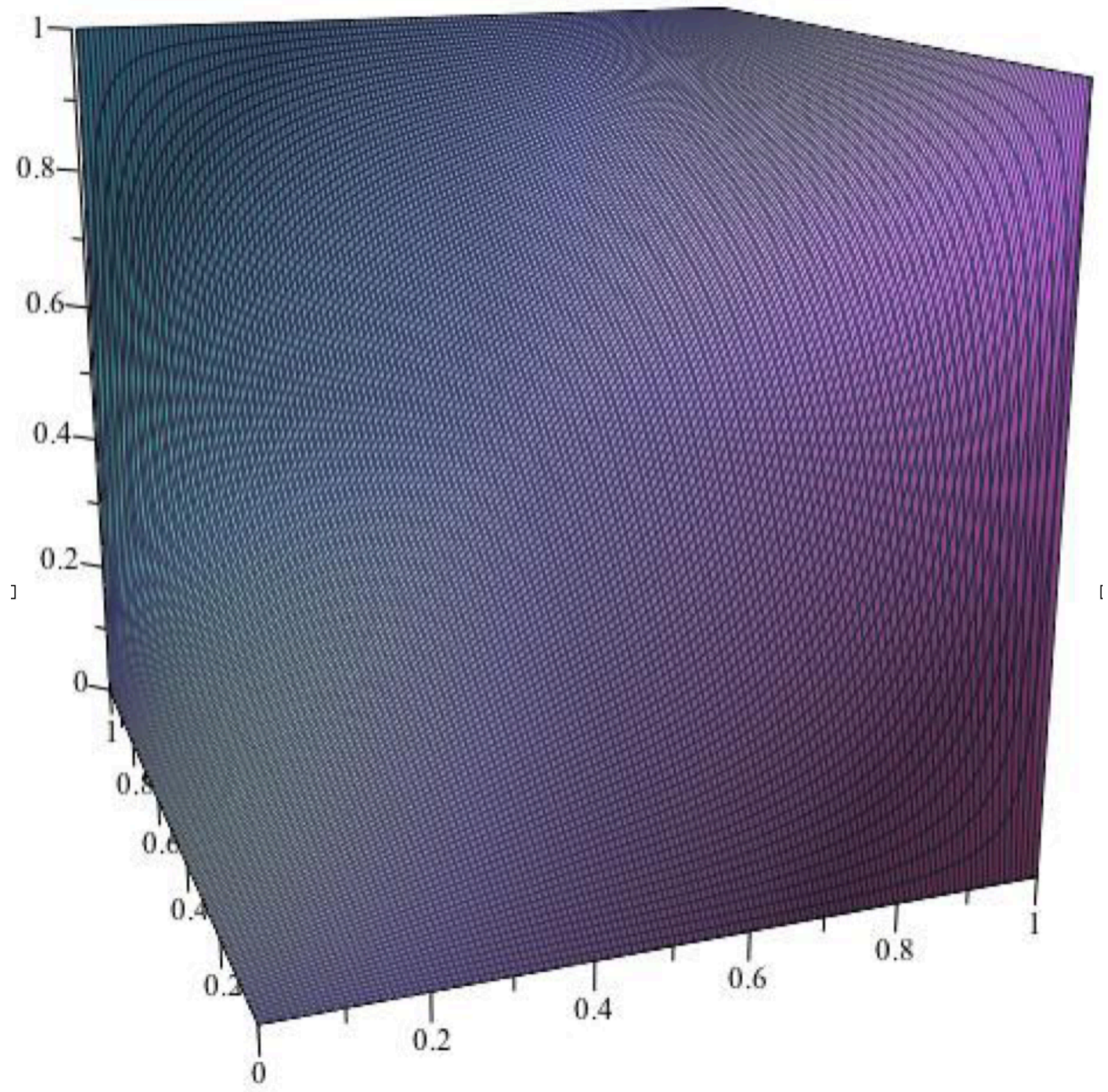


Fig. Visualization: The only ordered Abelian group over]0, 1[.

How to construct?

- $X \times Y$
- $X_1 \leq X \qquad (X_1 \times Y) \cup ((X \setminus X_1) \times \{.\})$
- $X_1 \leq X \qquad (X_1 \times Y^{\top}) \cup ((X \setminus X_1) \times \{\top\})$
- $X_1 \leq X \qquad (X_1 \times Y^{\top\perp}) \cup ((X \setminus X_1) \times \{\perp\})$
- Sufficient to generate densely-ordered algebras

How to construct?

- $X_2 \leq X_1 \leq X$ $\quad (X_1 \times Y^{\top}) \cup ((X \setminus X_1) \times \{\top\})$ $\quad \mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\top})}$

  $(X_2 \times Y) \cup (X_1 \times \{\top\}) \cup ((X \setminus X_1) \times \{\top\})$ $\quad \mathbf{X}^{\mathbf{\Gamma}(\mathbf{X_2}, \mathbf{Y})}_{\mathbf{\Gamma}(\mathbf{X_1}, \top)}$

- $X_2 \leq X_1 \leq X$ $\quad (X_1 \times Y^{\top\perp}) \cup ((X \setminus X_1) \times \{\perp\})$ $\quad \mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\top\perp})}$

  $(X_2 \times Y) \cup (X_1 \times \{\top, \perp\}) \cup ((X \setminus X_1) \times \{\perp\})$ $\quad \mathbf{X}^{\mathbf{\Gamma}(\mathbf{X_2}, \mathbf{Y})}_{\mathbf{\Gamma}(\mathbf{X_1}, \top\perp)}$

- Sufficient to generate densely-ordered algebras

- Sufficient to generate all algebras

How to construct? (details)

Definition *1*. Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \circledast, \to_\circledast, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $\overset{*}{'}$ and $'$, respectively.

1. Add a new element $\top$ to Y as a top element, and extend $\circledast$ by $\top \circledast y = y \circledast \top = \top$ for $y \in Y \cup \{\top\}$, then add a new element $\bot$ to $Y \cup \{\top\}$ as a bottom element, and extend $'$ by $\bot' = \top$, $\top' = \bot$ and $\circledast$ by $\bot \circledast y = y \circledast \bot = \bot$ for $y \in Y \cup \{\top, \bot\}$. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be cancellative subalgebras of $\mathbf{X}$ such that $\mathbf{X}_2 \leq \mathbf{X}_1$. We define $\mathbf{X}^{\mathbf{\Gamma}(\mathbf{X_2}, \mathbf{Y})}_{\mathbf{\Gamma}(\mathbf{X_1}, ^{\top\bot})} = \left(X^{\Gamma(X_2, Y)}_{\Gamma(X_1, ^{\top\bot})}, \leq, \diamond, \to_\diamond, (t_X, t_Y), (f_X, f_Y) \right)$, where

$$X^{\Gamma(X_2, Y)}_{\Gamma(X_1, ^{\top\bot})} = (X_1 \times \{\top, \bot\}) \cup (X_2 \times Y) \cup ((X \setminus X_1) \times \{\bot\}),$$

$\leq$ is the restriction of the lexicographical order of $\leq_X$ and $\leq_{Y \cup \{\top, \bot\}}$ to $X^{\Gamma(X_2, Y)}_{\Gamma(X_1, ^{\top\bot})}$, $\diamond$ is defined coordinatewise, and the operations $\overset{\diamond}{'}$ and $\to_\diamond$ are given by

$$(x_1, y_1) \to_\diamond (x_2, y_2) = \left((x_1, y_1) \diamond (x_2, y_2)^{\overset{\diamond}{'}} \right)^{\overset{\diamond}{'}} \text{ and } (x, y)^{\overset{\diamond}{'}} = \begin{cases} (x^{\overset{*}{'}}, y') & \text{if } x \in X_1 \\ (x^{\overset{*}{'}}, \bot) & \text{if } x \notin X_1 \end{cases}.$$

Call $\mathbf{X}^{\mathbf{\Gamma}(\mathbf{X_2}, \mathbf{Y})}_{\mathbf{\Gamma}(\mathbf{X_1}, ^{\top\bot})}$ the *(type III) partial-lexicographic product* of X, X_1, X_2, and Y.
In particular, if $\mathbf{X}_1 = \mathbf{X}_2$ then call $\mathbf{X}^{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y})}_{\mathbf{\Gamma}(\mathbf{X_1}, ^{\top\bot})}$ the *(type I) partial-lexicographic product* of X, X_1, and Y, and denote it by $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\top\bot})}$.

How to construct? (details)

Definition *1*. Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \circledast, \to_\circledast, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement ${}'^{*}$ and ${}'$, respectively.

2. Add a new element $\top$ to Y as a top element, and extend $\circledast$ by $\top \circledast y = y \circledast \top = \top$ for $y \in Y \cup \{\top\}$. Let $\mathbf{X}_1$ be a linearly ordered, discretely embedded[3], prime[4] and cancellative subalgebra of $\mathbf{X}$[5], and let $\mathbf{X}_2 \leq \mathbf{X}_1$. We define $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\top)}^{\mathbf{\Gamma}(\mathbf{X_2},\mathbf{Y})} = \left(X_{\Gamma(X_1,\top)}^{\Gamma(X_2,Y)}, \leq, \diamond, \to_\diamond, (t_X, t_Y), (f_X, f_Y)\right)$, where

$$X_{\Gamma(X_1,\top)}^{\Gamma(X_2,Y)} = (X_1 \times \{\top\}) \cup (X_2 \times Y) \cup ((X \setminus X_1) \times \{\top\})$$

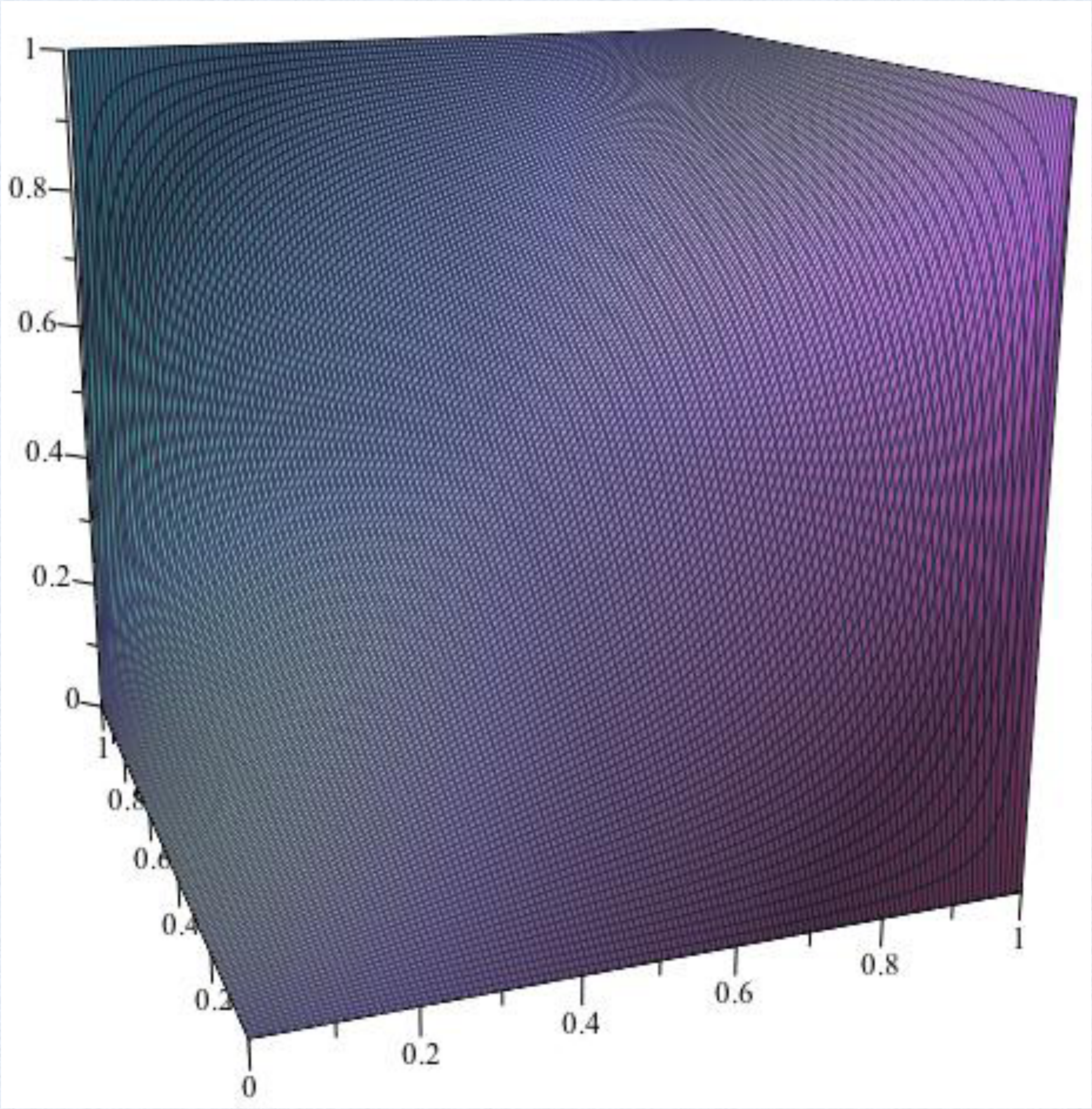
$\leq$ is the restriction of the lexicographical order of $\leq_X$ and $\leq_{Y \cup \{\top\}}$ to $X_{\Gamma(X_1,\top)}^{\Gamma(X_2,Y)}$, $\diamond$ is defined coordinatewise, and the operations ${}'^{\diamond}$ and $\to_\diamond$ are given by

$$(x_1, y_1) \to_\diamond (x_2, y_2) = \left((x_1, y_1) \diamond (x_2, y_2)'^{\diamond}\right)'^{\diamond} \quad \text{and} \quad (x, y)' = \begin{cases} (x'^{*}, \top) & \text{if } x \notin X_1 \text{ and } y = \top \\ (x'^{*}, y') & \text{if } x \in X_1 \text{ and } y \in Y \\ ((x'^{*})_\downarrow, \top) & \text{if } x \in X_1 \text{ and } y = \top \end{cases}.$$

Call $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\top)}^{\mathbf{\Gamma}(\mathbf{X_2},\mathbf{Y})}$ the *(type IV) partial-lexicographic product* of X, X_1, X_2, and Y.
In particular, if $\mathbf{X}_1 = \mathbf{X}_2$ then call $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\top)}^{\mathbf{\Gamma}(\mathbf{X_1},\mathbf{Y})}$ the *(type II) partial-lexicographic product* of X, X_1, and Y, and denote it by $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\mathbf{Y}^\top)}$.

It works!

Lemma 1. *Adopt the notation of Definition 1. Then* $\mathbf{X}_{\boldsymbol{\Gamma}(\mathbf{X_1},{}^{\top\perp})}^{\boldsymbol{\Gamma}(\mathbf{X_2},\mathbf{Y})}$ *and* $\mathbf{X}_{\boldsymbol{\Gamma}(\mathbf{X_1},{}^{\top})}^{\boldsymbol{\Gamma}(\mathbf{X_2},\mathbf{Y})}$ *are involutive* FL_e*-algebras with the same rank as that of* $\mathbf{Y}$.[6] $\mathbf{X}_{\boldsymbol{\Gamma}(\mathbf{X_1},{}^{\top\perp})}^{\boldsymbol{\Gamma}(\mathbf{X_2},\mathbf{Y})} \leq \mathbf{X}_{\boldsymbol{\Gamma}(\mathbf{X_1},\mathbf{Y}^{\top\perp})}$ *and* $\mathbf{X}_{\boldsymbol{\Gamma}(\mathbf{X_1},{}^{\top})}^{\boldsymbol{\Gamma}(\mathbf{X_2},\mathbf{Y})} \leq \mathbf{X}_{\boldsymbol{\Gamma}(\mathbf{X_1},\mathbf{Y}^{\top})}$ *hold. In particular, if* $\mathbf{Y}$ *is group-like then so are* $\mathbf{X}_{\boldsymbol{\Gamma}(\mathbf{X_1},{}^{\top\perp})}^{\boldsymbol{\Gamma}(\mathbf{X_2},\mathbf{Y})}$ *and* $\mathbf{X}_{\boldsymbol{\Gamma}(\mathbf{X_1},{}^{\top})}^{\boldsymbol{\Gamma}(\mathbf{X_2},\mathbf{Y})}$.



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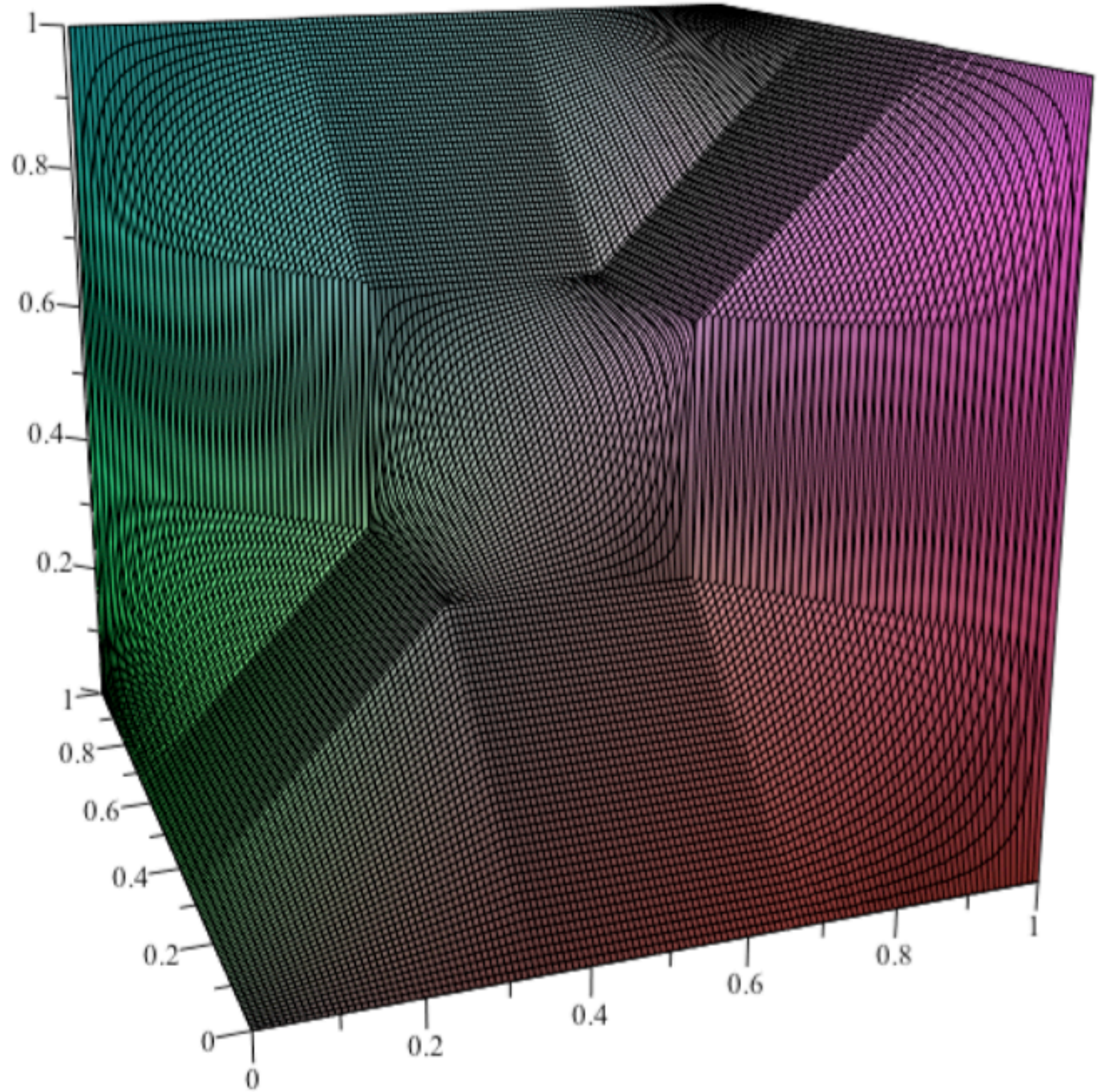
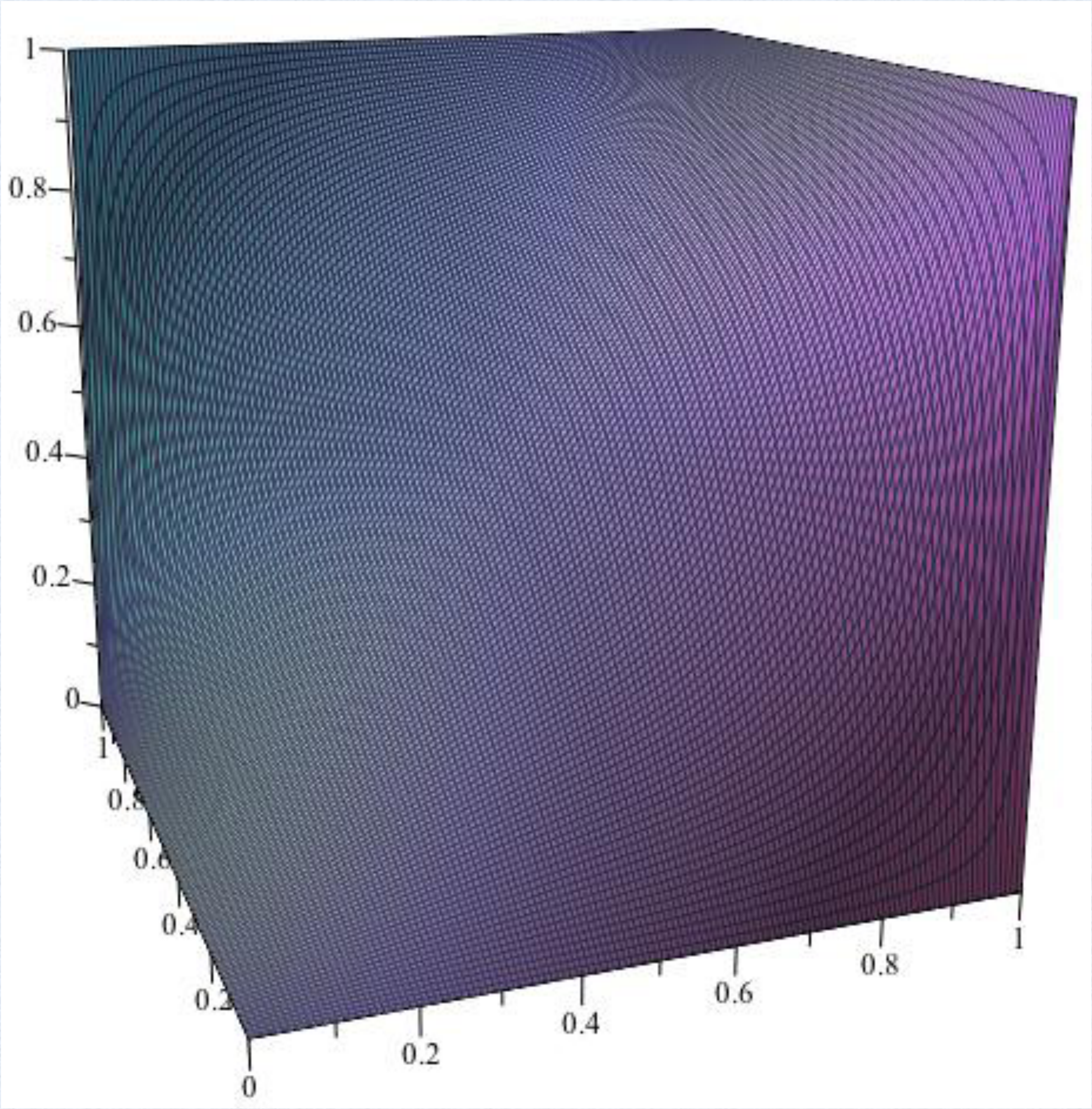


$$\mathbb{R}_{\mathbf{\Gamma}(\mathbf{1},\mathbb{R}^{\perp\top})}$$



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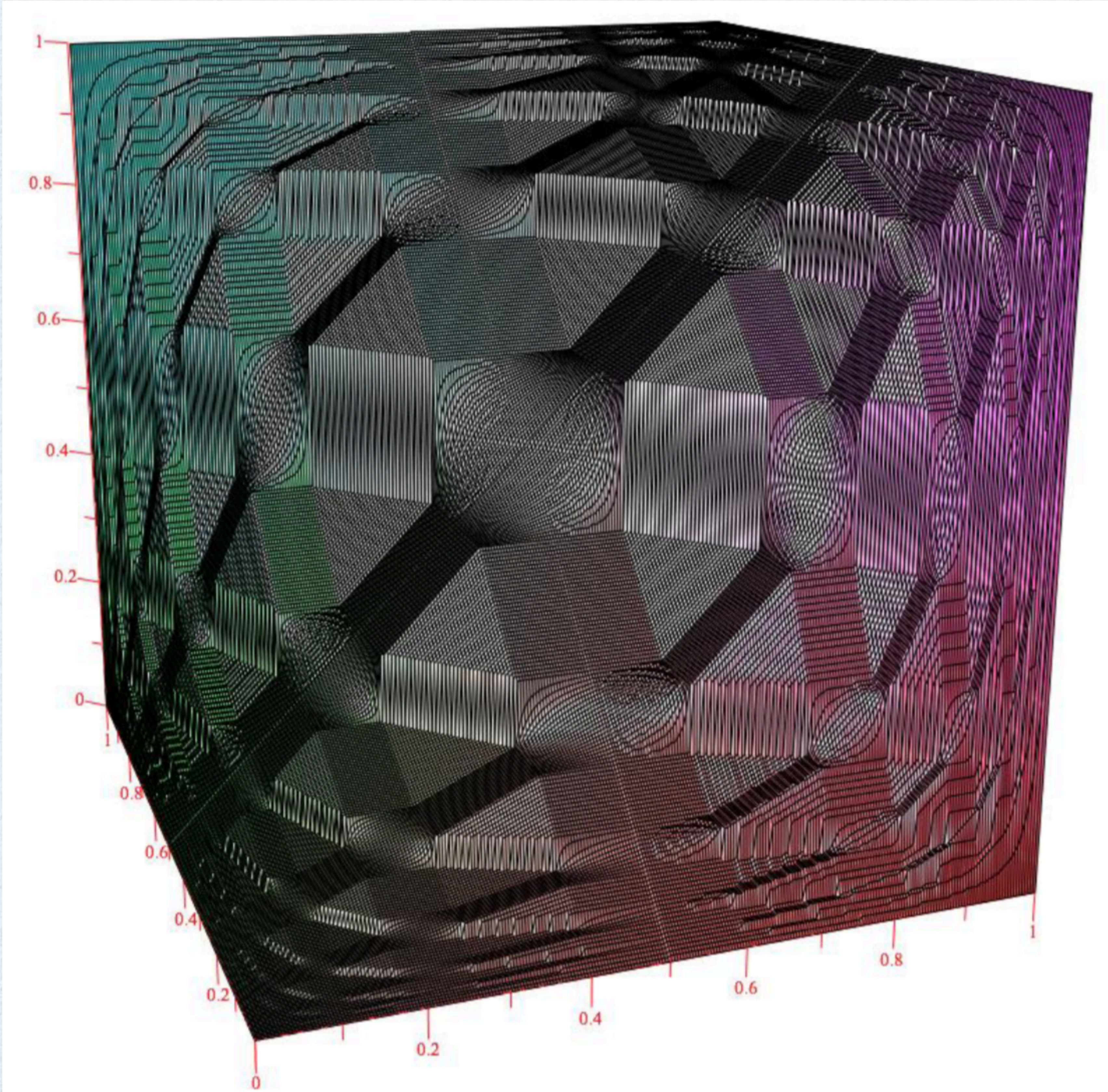
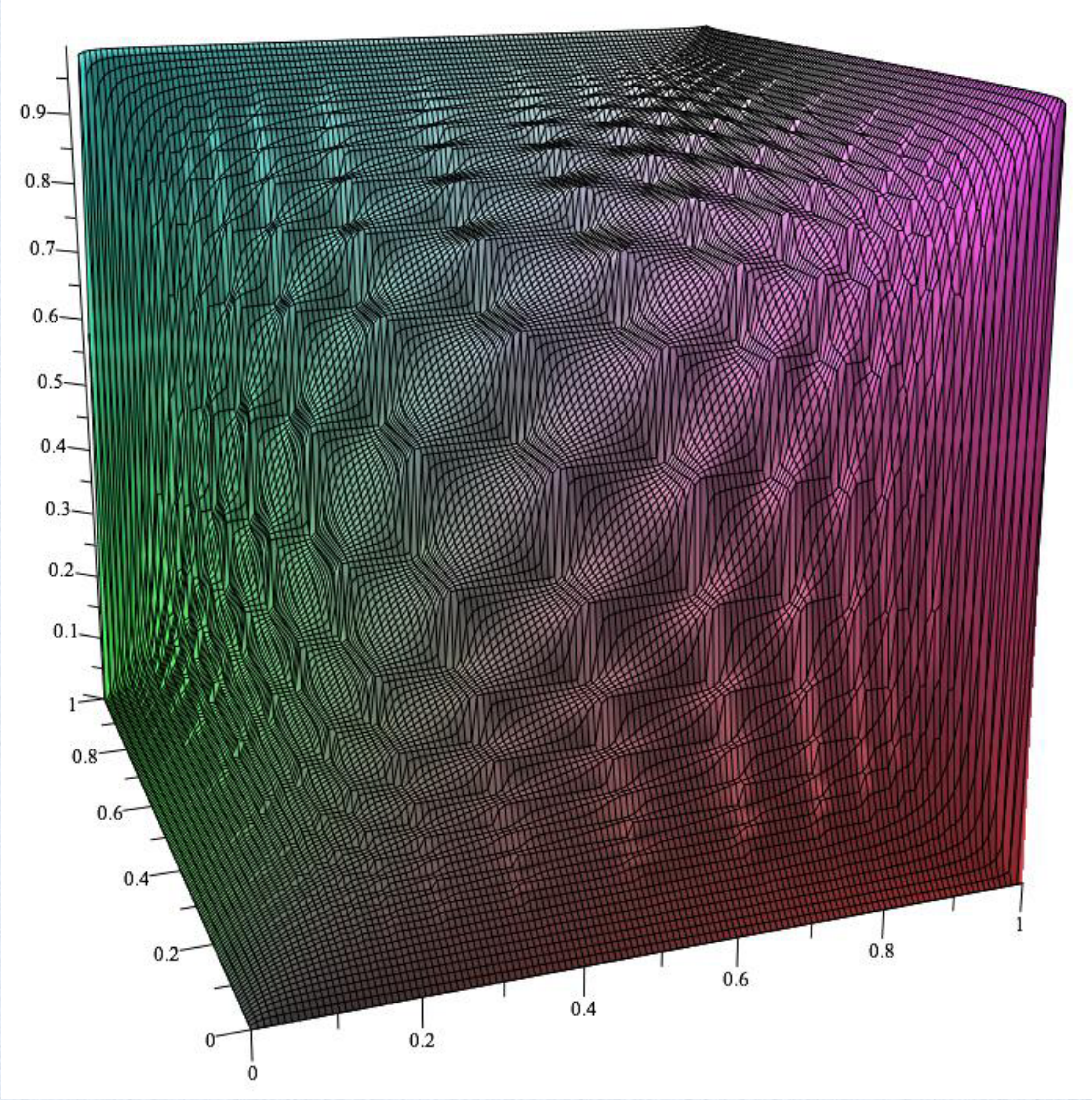


$\mathbb{R}_{\boldsymbol{\Gamma}(\mathbb{Z},\mathbb{R}^{\top\perp})}$



$\mathbb{Z}_{\mathbf{\Gamma}(\mathbb{Z},\mathbb{R}^\top)}$

Disconnected vs. Connected PLP construction

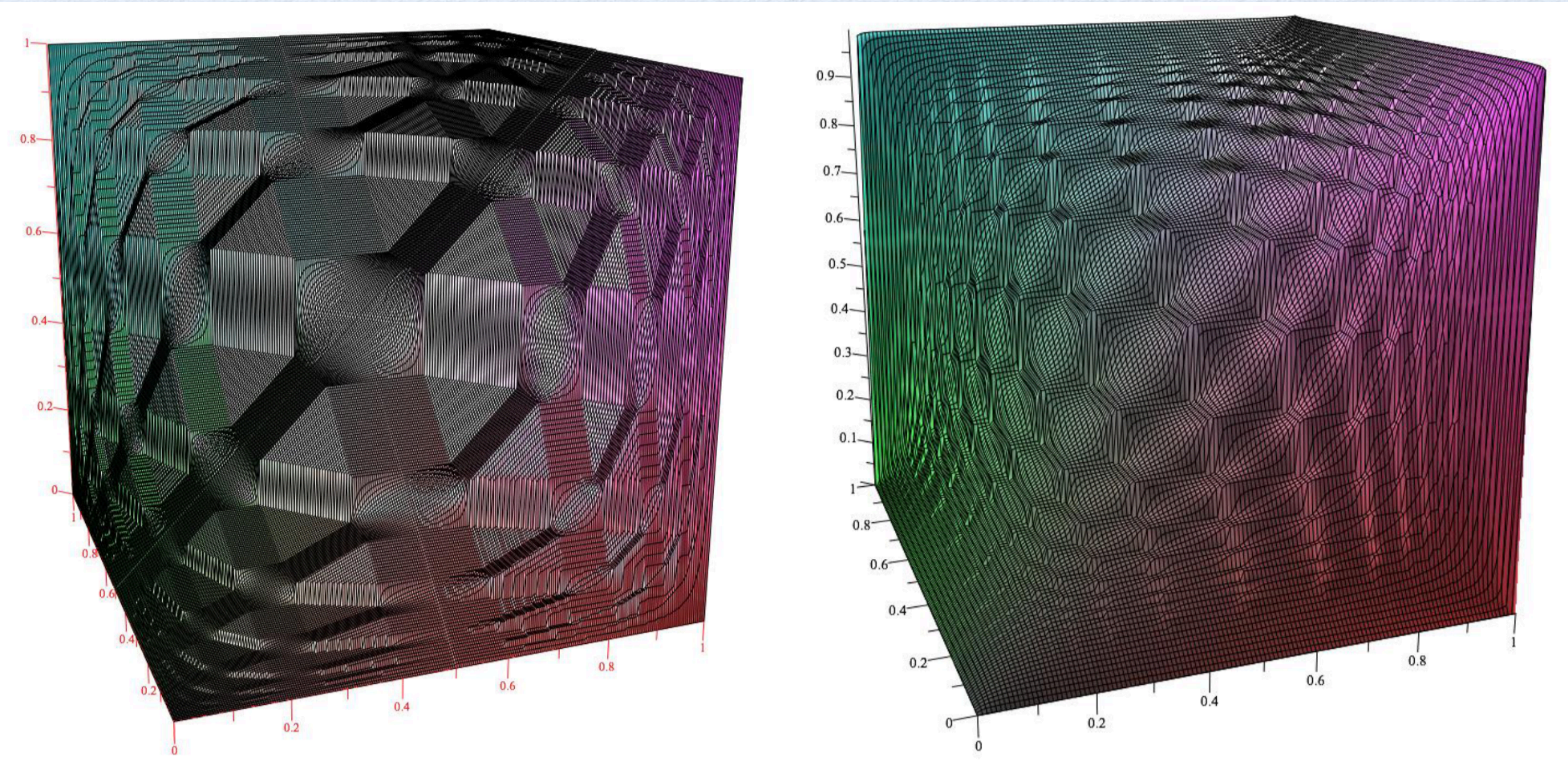


Fig. 1 Visualization: $\mathbb{R}_{\mathbf{\Gamma}(\mathbb{Z},\mathbb{R}^{\top\perp})}$ and $\mathbb{Z}_{\mathbf{\Gamma}(\mathbb{Z},\mathbb{R}^{\top})}$ shrank into $]0,1[$

Representation by totally ordered Abelian Groups

Theorem 2. (Group Representation) *If* $\mathbf{X}$ *is a group-like* FL_e*-chain, which has only* $n \in \mathbf{N}$, $n \geq 1$ *idempotents in its positive cone then there exist linearly ordered abelian groups* $\mathbf{G}_i$ $(i \in \{1, 2, \ldots, n\})$, $\mathbf{H}_{1,2} \leq \mathbf{H}_{1,1} \leq \mathbf{G}_1$, $\mathbf{H}_{i,2} \leq \mathbf{H}_{i,1} \leq \mathbf{\Gamma}(\mathbf{H}_{i-1,2}, \mathbf{G}_i)$ $(i \in \{2, \ldots, n-1\})$*, and a binary sequence* $\iota \in \{\top\bot, \top\}^{\{2,\ldots,n\}}$ *such that* $\mathbf{X} \simeq \mathbf{X}_n$*, where* $\mathbf{X}_1 := \mathbf{G}_1$ *and* $\mathbf{X}_i := \mathbf{X}_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1,1}, \iota_i)}^{\mathbf{\Gamma}(\mathbf{H}_{i-1,2}, \mathbf{G}_i)}$ $(i \in \{2, \ldots, n\})$.

Representation by totally ordered Abelian Groups

Theorem 2. **(Group Representation)** *If* $\mathbf{X}$ *is a group-like* FL_e*-chain, which has only* $n \in \mathbf{N}$, $n \geq 1$ *idempotents in its positive cone then there exist linearly ordered abelian groups* $\mathbf{G}_i$ $(i \in \{1, 2, \dots, n\})$, $\mathbf{H}_{1,2} \leq \mathbf{H}_{1,1} \leq \mathbf{G}_1$, $\mathbf{H}_{i,2} \leq \mathbf{H}_{i,1} \leq \mathbf{\Gamma}(\mathbf{H}_{i-1,2}, \mathbf{G}_i)$ $(i \in \{2, \dots, n-1\})$*, and a binary sequence* $\iota \in \{\top\bot, \top\}^{\{2,\dots,n\}}$ *such that* $\mathbf{X} \simeq \mathbf{X}_n$*, where* $\mathbf{X}_1 := \mathbf{G}_1$ *and* $\mathbf{X}_i := \mathbf{X}_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1,1},{}^{\iota_i})}^{\mathbf{\Gamma}(\mathbf{H}_{i-1,2},\mathbf{G}_i)}$ $(i \in \{2, \dots, n\})$.

Theorem 3. **(Structural description)** *If* $\mathbf{X}$ *is a densely ordered, group-like* FL_e*-chain, which has only* $n \in \mathbf{N}$ *idempotents in its positive cone then there exist linearly ordered Abelian groups* $\mathbf{G}_i$ $(i \in \{1, 2, \dots, n\})$, $\mathbf{H}_1 \leq \mathbf{G}_1$, $\mathbf{H}_i \leq \mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i)$ $(i \in \{2, \dots, n-1\})$*, and a binary sequence* $\iota \in \{\top\bot, \top\}^{\{2,\dots,n\}}$ *such that* $\mathbf{X} \simeq \mathbf{X}_n$*, where* $\mathbf{X}_1 := \mathbf{G}_1$ *and* $\mathbf{X}_i := \mathbf{X}_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1},\mathbf{G}_i{}^{\iota_i})}$ $(i \in \{2, \dots, n\})$. [24]

Representation by totally ordered Abelian Groups

Corollary 1. (Hahn-type embedding) *Group-like FL_e-chains with a finite number of idempotents embed in the finite partial-lexicographic product of lexicographic products of real groups.*

Corollary 2. (Lexicographical embedding of the monoid reduct) *The monoid reduct of any group-like FL_e-chain with a finite number of idempotents embeds in the* lexicographic *product of the 'extended' additive group of the reals*[28].

PARTIALLY ORDERED ALGEBRAIC SYSTEMS

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5. Hahn's embedding theorem

➢ This section is devoted to the deepest result in the theory of f. o. Abelian groups. This asserts the embeddability of f. o. Abelian groups in the lexicographic product of real groups.

HAHN, H. [1] Über die nichtarchimedischen Grössensysteme, *S.-B. Akad. Wiss. Wien. IIa,* **116** (1907), 601—655.

Comparison

- Hahn's theorem:
- Every totally ordered Abelian group embeds in a lexicographic product of real groups.

- Our embedding theorem:
- Every group-like FL_e-chain, which has finitely many idempotents embeds in a finite partial-lexicographic product of totally ordered Abelian groups.

An application in logic

§**2. Axiomatizations.** We base substructural fuzzy logics on a countable propositional language with formulas FOR built inductively as usual from a set of propositional variables VAR, binary connectives $\odot$, $\rightarrow$, $\wedge$, $\vee$, and constants $\bot$, $\top$, f, t, with defined connectives:

$$\begin{aligned} \neg A &=_{def} A \rightarrow f \\ A \oplus B &=_{def} \neg(\neg A \odot \neg B) \\ A \leftrightarrow B &=_{def} (A \rightarrow B) \wedge (B \rightarrow A) \end{aligned}$$

An application in logic

DEFINITION 1. **MAILL** *consists of the following axioms and rules:*

(L1) $A \to A$

(L2) $(A \to B) \to ((B \to C) \to (A \to C))$

(L3) $(A \to (B \to C)) \to (B \to (A \to C))$

(L4) $((A \odot B) \to C) \leftrightarrow (A \to (B \to C))$

(L5) $(A \wedge B) \to A$

(L6) $(A \wedge B) \to B$

(L7) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

(L8) $A \to (A \vee B)$

(L9) $B \to (A \vee B)$

(L10) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

(L11) $A \leftrightarrow (t \to A)$

(L12) $\bot \to A$

(L13) $A \to \top$

$$\frac{A \quad A \to B}{B}\ (mp) \qquad\qquad \frac{A \quad B}{A \wedge B}\ (adj)$$

An application in logic

DEFINITION 2. Uninorm logic **UL** *is* **MAILL** *plus:*

$$(PRL)\quad (A \to B) \wedge t) \vee ((B \to A) \wedge t)$$

- Involutive uninorm logic **IUL** *is* **UL** *plus* (INV) $\neg\neg A \to A$.
- Monoidal t-norm logic **MTL** *is* **UL** *plus* (W) $(A \to t) \wedge (f \to A)$.
- Involutive monoidal t-norm logic **IMTL** *is* **MTL** *plus* (INV).
- Gödel logic **G** *is* **MTL** *plus* (ID) $A \leftrightarrow (A \odot A)$.
- Uninorm mingle logic **UML** *is* **UL** *plus* (ID).
- Involutive uninorm mingle logic **IUML** *is* **IUL** *plus* (ID) *and* (FP) $t \leftrightarrow f$.

$\mathbf{IUL}^{fp}$ **IUL** plus $t \leftrightarrow f$

Finite Strong Standard Completeness

for any finite *theory* Γ *and a formula* φ:

1. $\Gamma \vdash_{\mathrm{L}} \varphi$.

2. $e(\varphi) = 1$ *for each standard* L-*algebra* $[0, 1]_*$ *and any* $*$*-model* e *of* Γ.

Finite Strong Standard Completeness

Theorem 4. *The logic* $\mathbf{IUL}^{fp}$ *enjoys finite strong standard completeness.*

That is all.