On some notions of negation between contraposition and minimal negation

Satoru Niki

School of Information Science, JAIST

July 22, 2018

Outline

Subminimal Logics

An$^-$PC and Kripke semantics

LPPC and Cut-elimination

Outline

Subminimal Logics

An^{-}PC and Kripke semantics

LPPC and Cut-elimination

Motivation

- Intuitionistic Logic
  - Classical Logic minus $(A \vee \neg A)$.
- Minimal Logic
  - Introduced in Johansson (1937).
  - Intuitionistic logic minus $(A \wedge \neg A) \rightarrow B$.
- In these logics, usually $\neg A := A \rightarrow \bot$.
- In minimal logic, $\bot$ has no axiom.

Motivation

Question

-Minimal logic validates $(A \wedge \neg A) \rightarrow \neg B$: *maybe too strong?*
-How to obtain a weaker negation when already no axiom for $\bot$?

Subminimal Logics

Colacito, de Jongh and Vargas (2017)

- Take $\neg$ as primitive, rather than $\bot$.
- Then M: $[(p \to q) \land (p \to \neg q)] \to \neg p$ defines minimal negation.
- Subminimal Logics: Logics with weaker negation axioms than M.

Subminimal Logics

Definition (Colacito et al.(2017))

Co: $(p \to q) \to (\neg q \to \neg p)$; NeF: $(p \wedge \neg p) \to \neg q$;
N: $(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$; An: $(p \to \neg p) \to \neg p$.

Proposition (Colacito (2016), Colacito et al.(2017))

(i) Co $\Rightarrow$ NeF and Co $\Rightarrow$ N
(ii) An+N $\Leftrightarrow$ M

I shall consider the following axioms.

Definition (N.)

LP: $(p \leftrightarrow \neg p) \to \neg p$; An^-: $(p \to \neg p) \to (\neg q \to \neg p)$;

Graphical Representation

Logic	*Negation Axiom(s)*
MPC$_\neg$	N + An: $(p \to \neg p) \to \neg p$
LPPC	N + An$^-$ + LP: $(p \leftrightarrow \neg p) \to \neg p$
An$^-$PC	N + An$^-$: $(p \to \neg p) \to (\neg q \to \neg p)$
CoPC	Co: $(p \to q) \to (\neg q \to \neg p)$
NeFPC	N + NeF: $(p \wedge \neg p) \to \neg q$
NPC	N: $(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$

Table

Logic	*proposed*	*Axiom(s)*	*Complete*	*Cut-elim.*
MPC$_\neg$	Johansson	M (N+An)	Yes	Yes
LPPC	N.	N+An$^-$+LP	Yes	?
An$^-$PC	N.	N+An$^-$	Yes	Yes
CoPC	Colacito et al.	Co	Yes	Yes
NeFPC	Colacito et al.	N+NeF	Yes	Yes
NPC	Colacito et al.	N	Yes	Yes

Outline

Subminimal Logics

An$^-$PC and Kripke semantics

LPPC and Cut-elimination

Languages

Definition ($\mathcal{L}_\neg, \mathcal{L}_{(\neg,\bot)}$)

We shall use the following propositional languages:

$$\mathcal{L}_\neg ::= p \mid \top \mid A \wedge B \mid A \vee B \mid A \to B \mid \neg A$$
$$\mathcal{L}_{(\neg,\bot)} ::= p \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid \neg A$$

NPC

We shall formulate **NPC** as a Hilbert system in $\mathcal{L}_\neg$.

Definition (NPC)

Positive Axioms

$p \to \top$; $p \to (q \to p)$; $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$;
$p \to (p \vee q)$; $q \to (p \vee q)$; $(p \to r) \to [(q \to r) \to ((p \vee q) \to r)]$;
$(p \wedge q) \to p$; $(p \wedge q) \to q$; $p \to (q \to (p \wedge q))$.

Negative Axioms

N: $(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$.

Rules

(MP): If $\Gamma \vdash A$ and $\Gamma \vdash A \to B$, then $\Gamma \vdash B$.
(Sub): If $\Gamma(p_1, \ldots p_n) \vdash A(p_1, \ldots p_n)$,
then $\Gamma(B_1 \ldots B_n) \vdash A(B_1 \ldots B_n)$.

An^-: A Weak An

Definition (An^-)

An^-: $(p \to \neg p) \to (\neg q \to \neg p)$

We define $\mathbf{An^-PC}$ as $\mathbf{NPC} + \mathrm{An}^-$.

Definition (Vakarelov (2005))

SUBMIN is a logic in $\mathcal{L}_{(\neg,\bot)}$ with the positive axioms&rules of **NPC**, and negative axioms Co, $\neg p \to \neg\neg\top$ and $\bot \to p$.
($\neg A$ is defined independent of $A \to \bot$.)

Proposition (N.)

$\mathbf{An^-PC}$ is equivalent to the $\mathcal{L}_\neg$-fragment of **SUBMIN**.

$\mathbf{An}^-\mathbf{PC}$ and Kripke Semantics

$\mathbf{An}^-\mathbf{PC}$ is complete with Vakalerov (2005)'s semantics.

Definition ((F, G)-semantics for $\mathbf{An}^-\mathbf{PC}$)

An (F, G)-frame is a quadruple $(W, \leq, F, G)$.

- $(W, \leq)$ is a poset.
- $F, G \subseteq W$ are upward closed subsets s.t. $F \subseteq G$;
- The valuation of negation is given by
  - $\mathcal{M}, w \Vdash \neg A \Leftrightarrow \forall w' \geq w[\mathcal{M}, w' \Vdash A \Rightarrow w' \in F] \wedge w \in G$

$\mathbf{An^-PC}$ and Kripke Semantics

$\mathbf{An^-PC}$ is complete with Vakalerov (2005)'s semantics.

Definition ((F, G)-semantics for $\mathbf{An^-PC}$)

An (F, G)-frame is a quadruple $(W, \leq, F, G)$.

- $(W, \leq)$ is a poset.
- $F, G \subseteq W$ are upward closed subsets s.t. $F \subseteq G$;
- The valuation of negation is given by
  - $\mathcal{M}, w \Vdash \neg A \Leftrightarrow \forall w' \geq w[\mathcal{M}, w' \Vdash A \Rightarrow w' \in F] \wedge w \in G$

- If we stipulate $G = W$, then the semantics is complete with minimal logic.
- If $G = W$ and $F = \emptyset$, then it is complete with intuitionistic logic.
- If $\forall w[w \in G \vee \exists w' \geq w(w' \in G \backslash F)]$ then complete with **LPPC**.

Subminimal Semantics

Colacito et al.(2017) gives Kripke semantics for subminimal logics.

Definition (N-semantics for **NPC**)

An N-frame for **NPC** is a triple $(W, \leq, N)$.

- $(W, \leq)$ is a poset.
- $N\colon \mathcal{U}(W) \to \mathcal{U}(W)$, where $\mathcal{U}(W)$ is the set of all upward closed subsets of W.
- N has to satisfy the next condition.
  - $w \in N(U) \Leftrightarrow w \in N(U \cap \mathcal{R}(w))$, where $\mathcal{R}(w) := \{w' | w' \geq w\}$.
- The valuation of negation is then defined as:
  - $w \Vdash \neg A \Leftrightarrow w \in N(\mathcal{V}(A))$, where $\mathcal{V}(A) := \{w | w \Vdash A\}$.

Subminimal Semantics

Colacito et al.(2017) gives Kripke semantics for subminimal logics.

Definition (N-semantics for **NPC**)

An N-frame for **NPC** is a triple $(W, \leq, N)$.

- $(W, \leq)$ is a poset.
- $N \colon \mathcal{U}(W) \to \mathcal{U}(W)$, where $\mathcal{U}(W)$ is the set of all upward closed subsets of W.
- N has to satisfy the next condition.
  - $w \in N(U) \Leftrightarrow w \in N(U \cap \mathcal{R}(w))$, where $\mathcal{R}(w) := \{w' | w' \geq w\}$.
- The valuation of negation is then defined as:
  - $w \Vdash \neg A \Leftrightarrow w \in N(\mathcal{V}(A))$, where $\mathcal{V}(A) := \{w | w \Vdash A\}$.

Semantics for other subminimal logics are obtained by suitably restricting N with additional conditions.

Relationship between the two semantics

Q. *How do the two semantics relate to each other?*

Definition (NAn$^-$-frame)

We call an N-frame with
$\forall U, V \in \mathcal{U}(W)[U \subseteq N(U) \Rightarrow N(V) \subseteq N(U)]$ as an NAn$^-$-frame.

Proposition (N.)

Let $\mathcal{F}$ be an N-frame. Then $\mathcal{F} \vDash_N \mathrm{An}^- \Leftrightarrow \mathcal{F}$ is an NAn$^-$-frame.

Relationship between the two semantics

Theorem (N.)

1. Let $\mathcal{F} = (W, \leq, N)$ be an $N\mathrm{An}^-$-frame. Then $\mathcal{F}' = (W, \leq, \bigcap_{U \in \mathcal{U}(W)} N(U), \bigcup_{U \in \mathcal{U}(W)} N(U))$ defines an (F, G)-frame such that $\mathcal{F} \models A \Leftrightarrow \mathcal{F}' \models A$.
2. Let $\mathcal{F} = (W, \leq, F, G)$ be an (F, G)-frame. Then $\mathcal{F}' = (W, \leq, N)$, s.t. $w \in N(U) \Leftrightarrow [\mathcal{R}(w) \cap U \subseteq F] \wedge w \in G$ defines an $N\mathrm{An}^-$-frame such that $\mathcal{F} \models A \Leftrightarrow \mathcal{F}' \models A$.

Hence the two semantics are equivalent over logics above **An$^-$PC**.

Outline

Subminimal Logics

$\mathbf{An^{-}PC}$ and Kripke semantics

LPPC and Cut-elimination

Axiom LP

Definition (LP)

LP: $(p \leftrightarrow \neg p) \to \neg p$

Definition (**LPPC**)

We define **LPPC** $:=$ $\mathbf{An^{-}PC} + \mathrm{LP}$

Axiom LP

Definition (LP)

LP: $(p \leftrightarrow \neg p) \rightarrow \neg p$

Definition (**LPPC**)

We define **LPPC** $:=$ **An**$^{-}$**PC** $+$ LP

Question

Is there a sequent calculus for **LPPC** wherein cut-elimination holds?

Axiom LP

Definition (LP)

LP: $(p \leftrightarrow \neg p) \to \neg p$

Definition (LPPC)

We define **LPPC** $:=$ **An**$^-$**PC** $+$ LP

Question

Is there a sequent calculus for **LPPC** wherein cut-elimination holds?

- Unsuccessful so far.
- Cut is eliminatable for **An**$^-$**PC** (using G3-type calculus).
- As a compromise, we shall prove cut-elimination for a different formulation of the logic.

Alternative Formulations of **LPPC**

We introduce variations of **LPPC** in order to prove cut-elimination.

Definition ($\mathcal{L}_2$, $\mathcal{L}_3$)

$\mathcal{L}_2 ::= p \mid \top \mid (A \wedge B) \mid (A \vee B) \mid (A \rightarrow B) \mid \neg A \mid \top^-$
$\mathcal{L}_3 ::= p \mid \top \mid (A \wedge B) \mid (A \vee B) \mid (A \rightarrow B) \mid \bot \mid \top^-$

Definition ($\mathbf{LP_2PC}$, $\mathbf{LP_3PC}$)

The logics $\mathbf{LP_2PC}$ and $\mathbf{LP_3PC}$ have the same positive axioms and rules as **LPPC**; they each have the following negative axioms.

$\mathbf{LP_2PC}$: N; $(p \rightarrow \neg p) \rightarrow (\top^- \rightarrow \neg p)$; $(p \leftrightarrow \neg p) \rightarrow \neg p$; $\neg p \rightarrow \top^-$.

$\mathbf{LP_3PC}$: $(\top^- \rightarrow \bot) \rightarrow \top^-$.

Alternative Formulations of **LPPC**

Theorem (N.)

1. $\mathbf{LP_2PC}$ is a conservative extension of **LPPC**.
2. There exist $t : \mathcal{L}_2 \to \mathcal{L}_3$ and $s : \mathcal{L}_3 \to \mathcal{L}_2$ such that $\vdash_{\mathbf{LP_2}} A \Leftrightarrow \vdash_{\mathbf{LP_3}} A^t$ and $\vdash_{\mathbf{LP_3}} A \Leftrightarrow \vdash_{\mathbf{LP_2}} A^s$.

- This justifies the view that $\mathbf{LP_3PC}$ is an alternative formulation of **LPPC**.
- We shall show cut-elimination for $\mathbf{LP_3PC}$.

Sequent Calculus for $\mathbf{LP_3PC}$

Definition ($\mathbf{GLP_3}$)

Axioms

Ax: $\Gamma, \varphi \Rightarrow \varphi$ (where $\varphi ::= p \mid \bot \mid \top^-$) R$\top$: $\Gamma \Rightarrow \top$

Rules for Positive Connectives

L$\wedge$: $\dfrac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}$ R$\wedge$: $\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$

L$\vee$: $\dfrac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$ R$\vee$: $\dfrac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}$ $(i \in \{1, 2\})$

L$\rightarrow$: $\dfrac{\Gamma, A \rightarrow B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C}$ R$\rightarrow$: $\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$

Rule for $\top^-$

$\top^-$: $\dfrac{\Gamma, \top^- \Rightarrow \bot}{\Gamma \Rightarrow \top^-}$

Cut-elimination for $\mathbf{LP_3PC}$

Definition (Cut)

Cut: $\dfrac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}$

We denote the addition of Cut to $\mathbf{GLP_3}$ as $\mathbf{GLP_3} + \text{Cut}$.

Proposition (equivalence with $\mathbf{LP_3PC}$ [N.])

$\Gamma \vdash_{\mathbf{LP_3}} A$ if and only if $\vdash_{\mathbf{GLP_3}+\text{Cut}} \Gamma \Rightarrow A$.

Cut-elimination for $\mathbf{LP_3PC}$

Definition (Cut)

Cut: $\dfrac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}$

We denote the addition of Cut to $\mathbf{GLP_3}$ as $\mathbf{GLP_3}$ + Cut.

Proposition (equivalence with $\mathbf{LP_3PC}$ [N.])

$\Gamma \vdash_{\mathbf{LP_3}} A$ if and only if $\vdash_{\mathbf{GLP_3}+\mathrm{Cut}} \Gamma \Rightarrow A$.

Theorem (cut-elimination for $\mathbf{GLP_3}$ [N.])

$\vdash_{\mathbf{GLP_3}} \Gamma \Rightarrow A$ if and only if $\vdash_{\mathbf{GLP_3}+\mathrm{Cut}} \Gamma \Rightarrow A$

Reference

- A. Colacito (2016). *Minimal and Subminimal Logic of Negation*. Master's Thesis. University of Amsterdam.
- A. Colacito, D. de Jongh and A.L. Vargas (2017). *Subminimal negation*. Soft Computing 21: 165-174.
- I. Johansson (1937), *Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus*. Compositio Mathematica 4:119-136.
- D. Vakarelov (1989). *Consistency, Completeness and Negation*, in G. Priest, R. Routley and J. Norma (eds.), *Paraconsistent Logic: Essays on the Inconsistent*. pp.328-363. Analityca, Philosophia Verlag.
- D. Vakarelov (2005). *Nelson's Negation on the Base of Weaker Versions of Intuitionistic Negation*. Studia Logica 80:393-430.