Combinatorial proofs and Genus of proofs

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July 24, 2018

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This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações) and Programa Doutoral em Matemática

Combinatorial Proofs

Problem of proofs identity

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash p, \bar{p}}\ \text{ax.}}{\vdash p, \bar{p}, q}\ \text{weak.} \qquad \dfrac{}{\vdash p, \bar{p}}\ \text{ax.}}{\vdash p \wedge p, \bar{p}, \bar{p}, q}\ \wedge}{\vdash p \wedge p, \bar{p}, q}\ \text{cont.} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash p, \bar{p}}\ \text{ax.} \qquad \dfrac{}{\vdash p, \bar{p}}\ \text{ax.}}{\vdash p \wedge p, \bar{p}, \bar{p}}\ \wedge}{\vdash p \wedge p, \bar{p}}\ \text{cont.}}{\vdash p \wedge p, \bar{p}, q}\ \text{weak.}$$

Combinatorial Proofs

Combinatorial proof is a mathematical formulation of propositional classical logic, in which proofs are graph-theoretic and combinatorial, instead of syntactic.

- graph of the formula
- coloured graph
- homomorphism

Combinatorial Proof-Graph formula

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- the vertices of G_ϕ are all the atomic occurrences in ϕ;
- the edges of G_ϕ are the defined as follows:
  - If ϕ is of the form $A \wedge B$ then G_ϕ is obtained by adding edges between any vertices of G_A and any of G_B;
  - No other edges appear in the graph.

Combinatorial Proof-Nicely Coloured Graph

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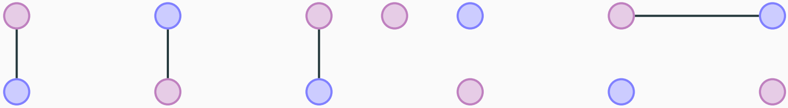
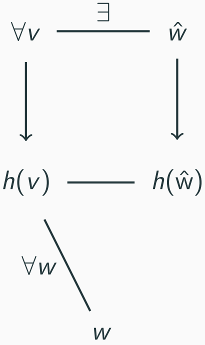
Example



Combinatorial Proof-Skew fibration

A graph morphism $h : G \to G'$ is a skew-fibration if for all $v \in V(G)$ and $(h(v), w) \in E(G')$ there exists $v\hat{w} \in E(G)$ s.t. $(h(\hat{w}), w) \notin E(G')$



Combinatorial Proof-Definition

A combinatorial proof of a proposition ϕ is a homomorphism $h : C \to G_\phi$ between two graphs s.t.:

- G_ϕ is the graph associated to the formula ϕ;
- C is a nicely coloured;
- C is axiomatic, that is for each two-vertex coloured class are labeled with dual propositional letters;
- h is a skew fibration.

Consider the Peirce's law:

$$((p \to q) \to p) \to p$$

$$((\overline{p} \vee q) \wedge \overline{p}) \vee p$$

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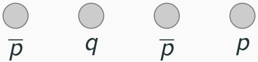
Now we build the graph of the formula.

First of all we need to consider the atoms, and those will be our nodes.

$$((\overline{p} \vee q) \wedge \overline{p}) \vee p$$



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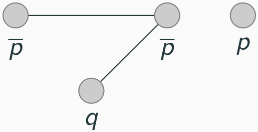

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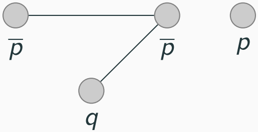
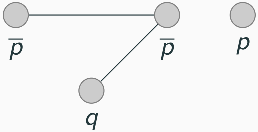


So we have the graph of the formula.

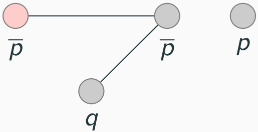
Coloured graph

Now we need to build the coloured graph associated to the graph of the formula.



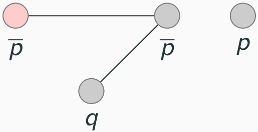
Coloured graph

Consider $\overline{p}$



Coloured graph

We need to find the dual



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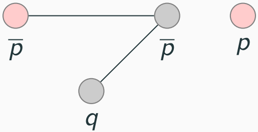
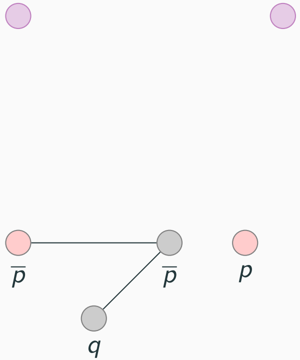


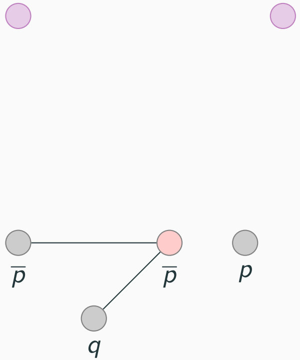
Coloured graph

So we can build the associate nodes in the coloured graph:



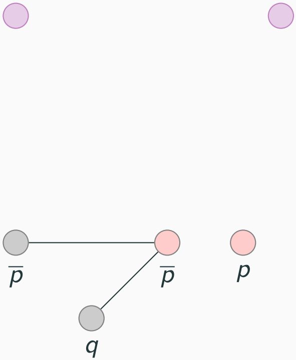
Coloured graph

Let's consider the second $\overline{p}$



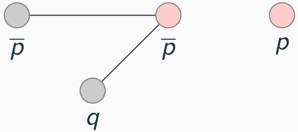
Coloured graph

Then we have:



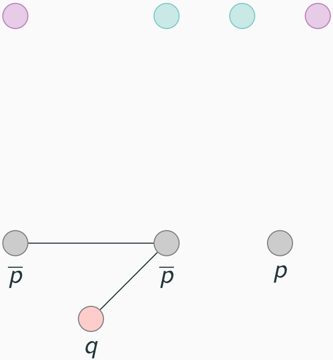
Coloured graph

And the associated nodes of the coloured graph:



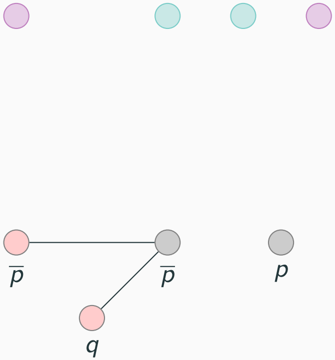
Coloured graph

And now for q:



Coloured graph

Let's look for $\overline{q}$:



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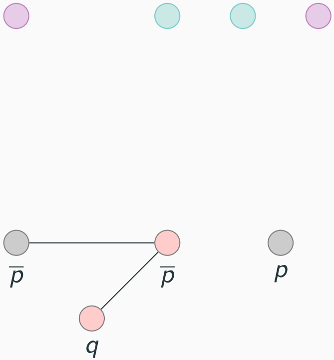

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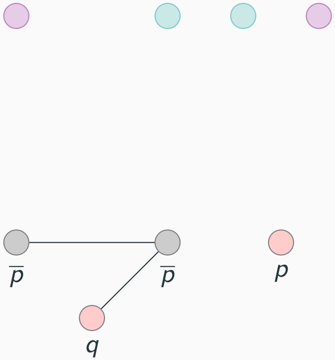
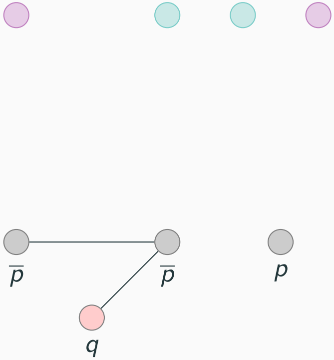


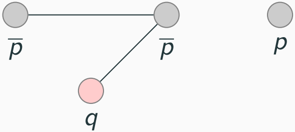
Coloured graph

Since we did not find the dual for q, we have no corresponded nodes in the coloured graph.



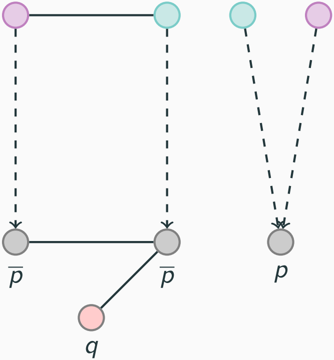
Coloured graph

Now the edges of the coloured graph:



Skew-Fibration, the final proof!

Now it is left to draw just the homomorphism:



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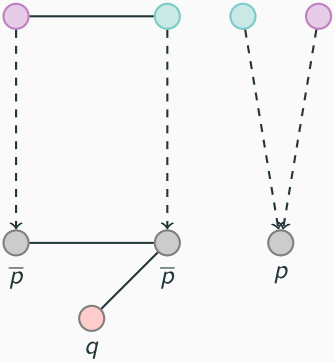

Is it a skew-fibration?

Mapping Sequent Calculus

$$\cfrac{\cfrac{\cfrac{\vdash p, \bar{p}}{\vdash p, \bar{p}, q}\ \text{weak.} \qquad \vdash p, \bar{p}}{\vdash p \wedge p, \bar{p}, \bar{p}, q}\ \wedge}{\vdash p \wedge p, \bar{p}, q}\ \text{con.}$$

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Genus

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Maximum number of non-intersecting cuts that can be made before the surface is disconnected.

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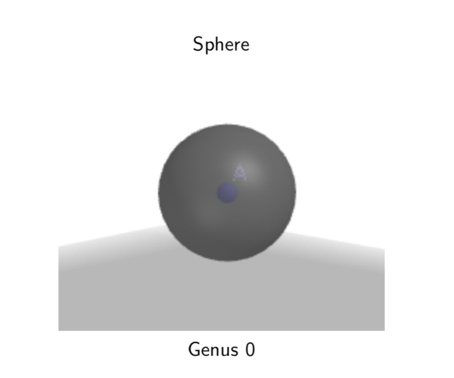

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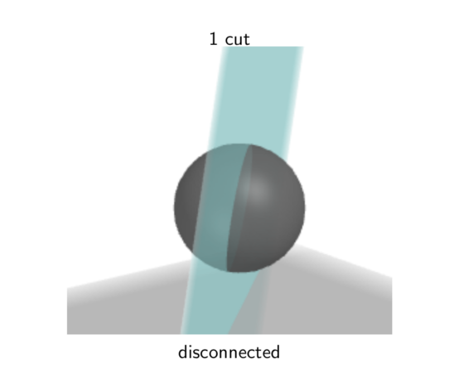

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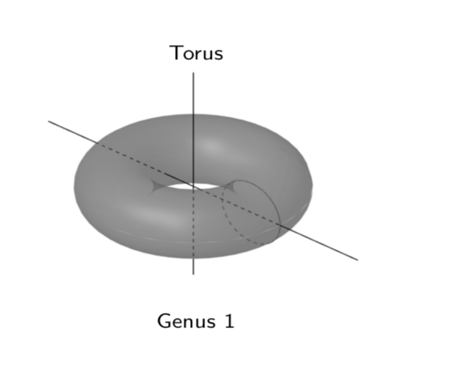

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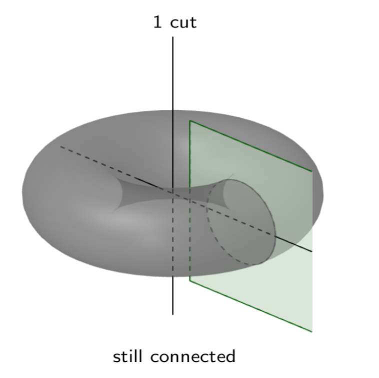

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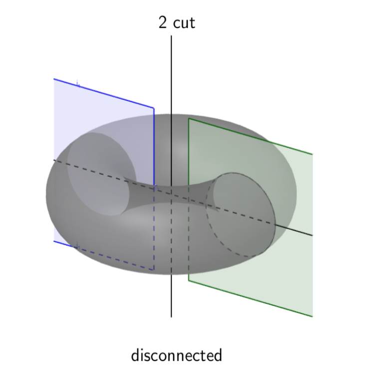
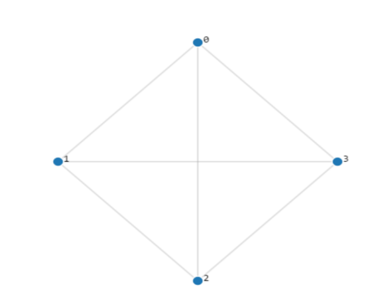


Genus of a graph:

The least genus of the surfaces which embed the graph without lines crossing.



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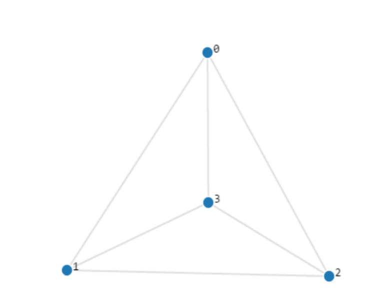

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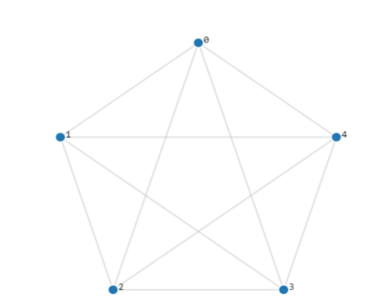

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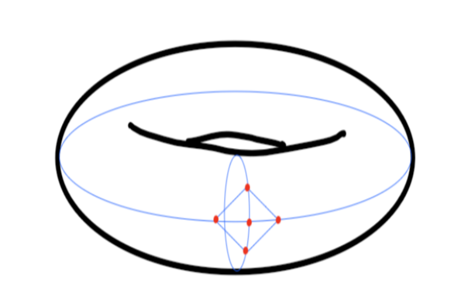

Genus in combinatorial proofs

Desiderata

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- Weakening does not influence complexity, i.e. it does not create new crossing.
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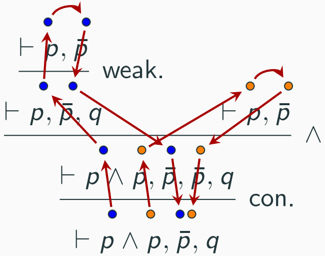
- When two different proofs generate the same combinatorial proofs then we want the mapping of combinatorial proof into the sequent derivations to have the same complexity.

(?) Cut and Contraction?

Genus in combinatorial proofs



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