Rogers semilattices in Ershov hierarchy

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Definition

For all $a \in \mathcal{O}$ set $A \subseteq \omega$ is Σ_a^{-1}-set if there are total computable functions $f(x,s)$ and $g(x,s)$ and for all $x \in \omega$:

1. $A(x) = \lim_s f(x,s)$, $f(x,0) = 0$, $g(x,0) = a$;
2. $g(x,s+1) \leq_o g(x,s)$
3. $f(x,s) \neq f(x,s+1) \to g(x,s+1) \neq g(x,s)$

Goncharov-Sorbi approach:

Definition
Numbering η *is called* Σ_a^{-1}*-computable, if set* $\{< x, y > | y \in \eta x\}$ *is a* Σ_a^{-1}*-set.*

Definition
$\eta \leqslant \mu$ *if there is computable function* f *and for all* $x \in \omega$

$\eta_x = \mu_{f(x)}$

For any family $\mathcal{S}$ of Σ_a^{-1}-sets, set $Com_a^{-1}(\mathcal{S})/_{\equiv}$ with $\leqslant$ forms upper semilattice $\mathcal{R}_a^{-1}$. We will call it Rogers semilattice.

Badaev, Manat, Sorbi 2012

For every nonzero $n \in \omega$, or $n = \omega$, and every notation $a \in \mathcal{O}$ of a nonzero ordinal there exists a Σ^{-1}_a- computable family $\mathcal{S}$ of cardinality n with $|\mathcal{R}^{-1}_a(\mathcal{S})| = 1$.

Questions:

1. What about other notations of this ordinal?
2. How stubborn this property could be?

Lemma

For any $a, b \in \omega$ $|a| = |b| < \omega^2$ *and any family of sets* $\mathcal{S}$ $\mathcal{R}_a^{-1}(\mathcal{S}) \cong \mathcal{R}_b^{-1}(\mathcal{S})$

Theorem

for any nonzero ordinals $\alpha, \beta < \omega^2$, *there are* $A, B \subseteq \omega$ *with* $|\mathcal{R}_\gamma^{-1}(\{A, B\})| = 1$ *for any* $max(\alpha, \beta) \leq \gamma < min(\alpha + \beta, \beta + \alpha)$.

Theorem
For any $a, b \in \mathcal{O}$, any $A \in \Sigma_a^{-1}$ and nonempty $B \in \Sigma_b^{-1}$, $\mathcal{R}_{2^{b+_O a}}^{-1}(\{A, B\})$ is infinite.

Corollary
For any notation $a \in \mathcal{O}$ of nonzero ordinal and any Σ_a^{-1}-computable family $\mathcal{S}$ $\mathcal{R}_{2^{a+_O a}(\mathcal{S})}^{-1}$ is infinite.

Theorem

There are $a, b \in \omega$, $|a| = |b| = \omega^2$ *and family* $\mathcal{S}$ *with* $\mathcal{R}_a^{-1}(\mathcal{S}) \not\cong \mathcal{R}_b^{-1}(\mathcal{S})$.

Theorem
There are $a, b \in \omega$, $|a| = |b| = \omega^2$ *and family* $\mathcal{S}$ *with* $\mathcal{R}_a^{-1}(\mathcal{S}) \not\cong \mathcal{R}_b^{-1}(\mathcal{S})$.

Proof idea.

Let $a, c \in \mathcal{O}$, $a <_O c$, $|a| = \omega^2$, $|c| = \omega^3$.

From Badaev-Manat-Sorbi we know that there is a family $\mathcal{S}$ with $|\mathcal{R}_a^{-1}(\mathcal{S})| = 1$ and infinite $\mathcal{R}_c^{-1}(\mathcal{S})$.

From Ershov we have that $\bigcup\limits_{a \in \mathcal{O}} \Sigma_a^{-1} = \bigcup\limits_{|a| = \omega^2} \Sigma_a^{-1}$

So, $\mathcal{S}$ also has infinite $\mathcal{R}_b^{-1}(\mathcal{S})$ for some $|b| = \omega^2$.

Thanks for your attention!

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