Strong compactness and the continuum function

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(Joint work in progress with A. Apter)

The continuum function

Definition

The continuum function is given by

$$\kappa \mapsto 2^\kappa$$

where κ is any infinite cardinal.

- $\kappa^+ \leqslant 2^\kappa$
- $\kappa \leqslant \lambda \rightarrow 2^\kappa \leqslant 2^\lambda$
- $\mathrm{cf}(2^\kappa) > \kappa$ (König's theorem)

GCH

Definition

The Generalised Continuum Hypothesis (GCH) is the statement that for all infinite cardinals κ, $2^\kappa = \kappa^+$.

- GCH asserts that the continuum function is as simple as possible.

Theorem (Gödel-1938 & Cohen-1963)

GCH is independent from ZFC.

Failures of GCH

Theorem (Easton, 1970)

Suppose F is an Easton function, i.e. a class function with domain the class of infinite regular cardinals, codomain the class of infinite cardinals and for all κ, λ in its domain:

1. $\kappa^+ \leqslant F(\kappa)$
2. $\kappa \leqslant \lambda \rightarrow F(\kappa) \leqslant F(\lambda)$
3. $\mathrm{cf}(F(\kappa)) > \kappa$

Then, there is a model of ZFC where F is realised, i.e. the continuum function on regular cardinals is identical to F.

- ZFC does not impose any restrictions on the continuum function on regular cardinals.

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- (Scott, 19??) A measurable cardinal cannot be the first failure of GCH.
- (Solovay, 1974) GCH holds at all singular cardinals above a strongly compact cardinal.
- (Folklore) If GCH holds below a supercompact cardinal, then it holds above it too.

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This shows that Easton's model may not admit large cardinals.

Question

What sort of Easton functions can be realised in the presence of large cardinals?

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Suppose κ is a measurable cardinal.

1. The continuum function can behave arbitrarily above κ. [Folklore?]
2. The continuum function can behave arbitrarily at a measure 0 set below κ. [Kunen & Paris, 1971]
3. The violation of GCH at a measurable cardinal is *not* provable just assuming a measurable. In fact, a measurable with non-trivial Mitchell rank is needed. [Mitchel-1984 & Gitik-1989]

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Theorem (Friedman & Honzik, 2008)

If κ is measurable, then we can realise any Easton function F, while preserving the measurability of κ, as long as:

1. $\forall \alpha < \kappa \ (F(\alpha) < \kappa)$,
2. *there is an embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $H_{F(\kappa)} \subseteq M$ and $j(F)(\kappa) \geqslant F(\kappa)$.*

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Theorem (Menas, 1976)

If κ is supercompact and F is a Δ_2-definable Easton function, then it is possible to realise F while preserving the supercompactness of κ.

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Theorem (Menas, 1976)

If κ is supercompact and F is a Δ_2-definable Easton function, then it is possible to realise F while preserving the supercompactness of κ.

Theorem (Cody, Friedman & Honzik, 2013)

If κ is supercompact and F is an Easton function with the additional properties:

1. $\forall \alpha < \kappa \; (F(\alpha) < \kappa)$,
2. *there is an embedding* $j : V \to M$ *with* $\mathrm{crit}(j) = \kappa$, ${}^{\lambda}M \subseteq M$, $H_{F(\lambda)} \subseteq M$ *and* $j(F) \restriction (\lambda + 1) = F \restriction (\lambda + 1)$,

then F can be realised while preserving the λ-supercompactness of κ.

The case of strongly compact cardinals

- In joint work in progress with A. Apter, we try to find what configurations in Easton's theorem would be compatible with strongly compact cardinals.
- It is not known if we can violate GCH at a strongly compact without stronger assumptions.

New results

Theorem (D., 2018, maybe known?)

If there is a cardinal κ which is strongly compact and κ^+-supercompact, then it is consistent that GCH fails at a strongly compact cardinal.

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Proof sketch.

- Start with an embedding $j : V \to M$ which witnesses both the κ^+-supercompactness and some degree of strong compactness.
- Factor the embedding through a κ^+-supercompact embedding.
- The κ^+-supercompactness embedding can be lifted using Magidor's argument (or surgery).
- The factor embedding has small width so we can also lift it. □

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Theorem (Magidor-unpublished, Apter-1998)

There is a model of ZFC where the classes of supercompact and strongly compact cardinals coincide whenever that is possible.

- Call a model as above “ideal”.

New results

Theorem (Apter, 2017)

In an "ideal" model, if κ is a non-supercompact strongly compact cardinal, it is possible to violate GCH at κ.

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Proof sketch.

- κ has to be a limit of supercompacts - make them all indestructible.
- Add a fast function for the **tallness** of κ.
- Force with the lottery preparation with respect to the fast function.
- Change the continuum at κ, using surgery to preserve its tallness. □

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Conjecture

*In an "ideal" model, it is possible to violate GCH at *all* strongly compact cardinals, while preserving their strong compactess.*

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Thank you!