Epistemic logic of sequences of partitions

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Objective

1. Modal system *OPS5*;

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2. $OPS5$ system and sequences of refinement of orthopairs;

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2. $OPS5$ system and sequences of refinement of orthopairs;
3. Epistemic interpretation of $OPS5$.

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*OPS*5-language

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- a set $\mathcal{V}$ of propositional variables;
- the logical connectives $\wedge, \neg$ and $\top$;
- the families of modal operators $\{\Box_i\}_{i\in I}$, $\{\triangle_i\}_{i\in I}$ and $\{\triangleright_i\}_{i\in I}$.

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Other connectives and well formed formulae are defined as usual.

*OPS*5 modal system

Axioms of *OPS*5

Axioms of the propositional logic + the following axioms:

1 ϕ is a tautology in propositional logic

2 $\Box_i\varphi \equiv \triangle_i\neg\varphi$

3 $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$

5 $\varphi \to \Box_i\neg\triangle_i\varphi$

6 $\Box_i\varphi \to \Box_i\Box_i\varphi$

7 $\rhd_i\top \to \Box_{i+1}\top$

8 $(\neg \rhd_i \varphi \wedge \varphi) \to \neg \rhd_{i+1} \varphi$

9 $\rhd_i\varphi \to (\Box_j\varphi \to \Box_{i+1}\varphi)$, with $j \leq i$

10 $\rhd_i\varphi \to \rhd_j\varphi$, with $1 < j \leq i$

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3 $\Box_i(\varphi \to \psi) \to (\Box_i \varphi \to \Box_i \psi)$
5 $\varphi \to \Box_i \neg \triangle_i \varphi$
6 $\Box_i \varphi \to \Box_i \Box_i \varphi$
7 $\rhd_i \top \to \Box_{i+1} \top$
8 $(\neg \rhd_i \varphi \wedge \varphi) \to \neg \rhd_{i+1} \varphi$
9 $\rhd_i \varphi \to (\Box_j \varphi \to \Box_{i+1} \varphi),$ with $j \leq i$
10 $\rhd_i \varphi \to \rhd_j \varphi,$ with $1 < j \leq i$

Inference rules

- Modus ponens;
- if $\varphi \to \psi$, then $\vdash \Box_i \varphi \to \Box_i \psi$ for each $i \in I$.

*OPS*5 modal system

Semantics of *OPS*5

A Kripke model for *OPS*5 is a triple $\mathcal{M} = (U, (R_1, \ldots, R_n), v)$, where

1. U is the universe of possible worlds,

$OPS5$ modal system

Semantics of $OPS5$

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1. U is the universe of possible worlds,
2. $(R_1, \ldots, R_n)$ is a sequence of equivalence relations of U such that $R_{i+1}(u) \subseteq R_i(u)$ for each $u \in U$ and $i < n$,

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3. v is the evaluation function such that
   - $\mathcal{M}, u \models \Box_i \varphi$ iff $R_i(u) \subseteq v(\varphi)$ and $R_i(u) \neq \{u\}$,
   - $\mathcal{M}, u \models \triangle_i \varphi$ iff $R_i(u) \cap v(\varphi) = \emptyset$ and $R_i(u) \neq \{u\}$,
   - $\mathcal{M}, u \models \triangleright_i \varphi$ iff $u \models \varphi$ and $R_i(u) \neq \{u\}$, with $i < n$,

where $i \in I$, $\varphi \in Form$ and $v(\varphi) = \{u \in U \mid \mathcal{M}, u \models \varphi\}$.

*OPS*5 modal system

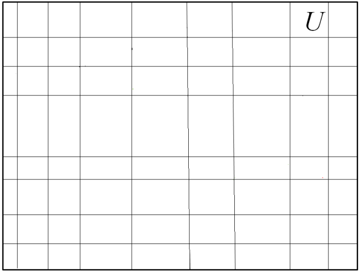
Theorem

The modal system OPS5 is

- *sound*,
- *consistent and*
- *complete.*

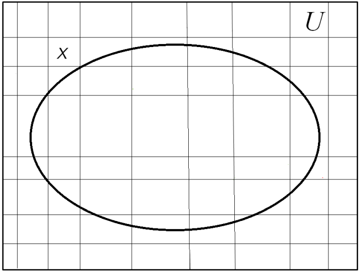
Orthopairs

Let P be a partition of U.



Orthopairs

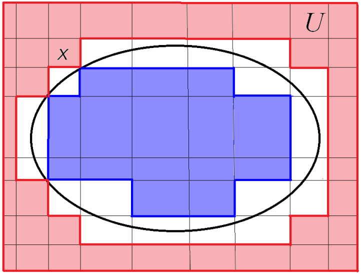
Let P be a partition of U and $X \subseteq U$.



Orthopairs

The orthopair of X determined by P is the pair

$$(\mathcal{L}_P(X), \mathcal{E}_P(X)) = (\textit{lower approximation}, \textit{impossibility domain})$$



Refinement sequences

Definition

A sequence $\mathcal{P} = P_1, \ldots, P_n$ of partial partitions of U is a refinement sequence of partial partitions of U if each element of P_i is contained in an element of P_{i-1}, for $i = 2, \ldots, n$.

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Example

Let $U = \{a, b, c, d, e, f, g, h, i\}$ and

- $P_1 = \{\{a, b, c, d, e, f, g, h, i\}\}$;
- $P_2 = \{\{a, b, c, d\}, \{e, f, g, h\}\}$(missing i);
- $P_3 = \{\{a, b\}, \{c, d\}, \{e, f, g\}\}$ (missing h).

(P_1, P_2, P_3) is a refinement sequence of U.

We consider the refinement sequences such that each block of each partition is not a singleton.

Sequence of Refinements of Orthopairs

Definition

Let $\mathcal{P} = P_1, \ldots, P_n$ be a refinement sequence of U. For every $X \subseteq U$, $\mathcal{P}$ determines the sequence of refinement of orthopairs

$$((\mathcal{L}_1(X), \mathcal{E}_1(X)), \ldots, (\mathcal{L}_n(X), \mathcal{E}_n(X)).$$

Remark

The sequences of refinement of orthopairs are a concrete representation of the finite IUML-algebras.

"Aguzzoli, S., Boffa, S., Ciucci, D., Gerla, B. (2016, October). Refinements of Orthopairs and IUML-algebras. In International Joint Conference on Rough Sets (pp. 87-96). Springer International Publishing."

OPS5 and Sequence of Refinement of Orthopairs

Theorem

Let $\mathcal{M} = (U, (R_1, \ldots, R_n), v)$ be a Kripke model of OPS5.

Then, for each $i \in I$

- *$P_1, \ldots, P_n$ is a refinement sequence of U, where $P_i = \{R_i(u) \ : \ u \in U \text{ and } R_i(u) \neq \{u\}\}$.*

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- *$v(\Box_i \varphi) = \mathcal{L}_i(v(\varphi))$ and $v(\triangle_i \varphi) = \mathcal{E}_i(v(\varphi))$, for each $\varphi \in Form$.*

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- *$v(\Box_i \varphi) = \mathcal{L}_i(v(\varphi))$ and $v(\triangle_i \varphi) = \mathcal{E}_i(v(\varphi))$, for each $\varphi \in Form$.*
- *$\rhd_i$ take care of the elements that survive in partition P_i. If $\mathcal{M}, u \models \rhd_i \varphi$, then $u \in \cup\{b \mid b \in P_i\}$.*

$OPS5$ as an Epistemic Logic

$OPS5$ allows to capture the knowledge of an agent $\mathcal{K}$ during a sequence $(t_1, \ldots, t_n)$ of successive instants of time.

$$(\mathcal{K}, (t_1, \ldots, t_n)) \quad \longleftrightarrow \quad (U, (R_1, \ldots, R_n), v)$$

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Possible words

For each $i \in \{1, \ldots, n\}$, let $u \in U$

- If $R_i(u) = \{u\}$, then "at time t_i, $\mathcal{K}$ ignores u",
- If $R_i(u) \neq \{u\}$, then "at time t_i, $\mathcal{K}$ is not able to distinguish the worlds of $R_i(u)$ from each other".

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Interpretation of $\Box_i$ and $\rhd_i$

- $u \models \Box_i \varphi$ is read "at time t_i, $\mathcal{K}$ knows that φ is true in u",
- $u \models \rhd_i \varphi$ is read "φ is true in u and at time t_i $\mathcal{K}$ is interested to know it ".

Example

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Every help from Bob to Alice determines an equivalence relation of the Alice's Kripe model.

ij is the card with number or figure i and suit j.

Time t_1

$\{ij \in \mathcal{D} \mid color(ij) = \textbf{red}\}$ $\{ij \in \mathcal{D} \mid color(ij) = \textbf{black}\}$

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First equivalence relation: $ij \; R_1 \; lm \quad \Leftrightarrow \quad color(ij) = color(lm)$.

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$\varphi =$ " the card is red"

$$R_1(2\diamondsuit) = \{ij \in \mathcal{D} : color(ij) = \textbf{red}\} = v(\varphi)$$

$$R_1(2\diamondsuit) \subseteq v(\varphi) \quad \text{and} \quad R_1(2\diamondsuit) \neq \{2\diamondsuit\}$$

$$\Updownarrow$$

$$2\diamondsuit \models \Box_1 \varphi$$

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"At time t_1, Alice knows that $2\diamondsuit$ is red"

Time t_1

$\varphi' =$ " the card is a two"

$v(\varphi') = \{2\diamondsuit, 2\heartsuit, 2\spadesuit, 2\clubsuit\}$

$R_1(2\diamondsuit) \nsubseteq v(\varphi')$

$\Downarrow$

$2\diamondsuit \not\models \Box_1 \varphi'$

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$$R_1(2\diamondsuit) \not\subseteq v(\varphi')$$

$$\Downarrow$$

$$2\diamondsuit \not\models \Box_1 \varphi'$$

"At time t_1, Alice does not know that $2\diamondsuit$ is a two"

Time $t_2 > t_1$

$\{i\diamondsuit \mid i \in \{1, \ldots, 10\}\}$ $\{i\heartsuit \mid i \in \{1, \ldots, 10\}\}$ $\{i\spadesuit \mid i \in \{1, \ldots, 10\}\}$ $\{i\clubsuit \mid i \in \{1, \ldots, 10\}\}$

Time $t_2 > t_1$

$\{i\diamondsuit \mid i \in \{1,\dots,10\}\}$ $\{i\heartsuit \mid i \in \{1,\dots,10\}\}$ $\{i\spadesuit \mid i \in \{1,\dots,10\}\}$ $\{i\clubsuit \mid i \in \{1,\dots,10\}\}$

Second equivalence relation:

$$ij\ R_2\ lm \ = \begin{cases} j = m, & \text{if } i, l \notin \{J, Q, K\} \\ ij = lm, & \text{otherwise} \end{cases}$$

$$R_2(J\diamondsuit) = \{J\diamondsuit\}, \quad R_2(Q\diamondsuit) = \{Q\diamondsuit\} \quad R_2(K\diamondsuit) = \{K\diamondsuit\}$$
$$R_2(J\heartsuit) = \{J\heartsuit\}, \quad R_2(Q\heartsuit) = \{Q\heartsuit\} \quad R_2(K\heartsuit) = \{K\heartsuit\}$$
$$R_2(J\spadesuit) = \{J\spadesuit\}, \quad R_2(Q\spadesuit) = \{Q\spadesuit\} \quad R_2(K\spadesuit) = \{K\spadesuit\}$$
$$R_2(J\clubsuit) = \{J\clubsuit\}, \quad R_2(Q\clubsuit) = \{Q\clubsuit\} \quad R_2(K\clubsuit) = \{K\clubsuit\}$$

$\psi =$ "the suit of the card is heart"

$$v(\psi) = \{i\heartsuit \ : \ i \in \{1, \ldots, 10\} \cup \{J, Q, K\}\}$$

$$R_2(5\heartsuit) = \{i\heartsuit \ : \ i \in \{1, \ldots, 10\}\}$$

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$$R_2(5\heartsuit) \subseteq v(\psi) \quad \text{and} \quad R_2(5\heartsuit) \neq \{5\heartsuit\}$$

$$\Updownarrow$$

$$5\heartsuit \models \Box_2 \psi$$

"At time t_2, Alice knows that the suit of $5\heartsuit$ is heart".

$\psi =$ "the suit of the card is heart"

$$v(\psi) = \{i\heartsuit \ : \ i \in \{1, \dots, 10\} \cup \{J, Q, K\}\}$$

$$R_2(5\heartsuit) = \{i\heartsuit \ : \ i \in \{1, \dots, 10\}\}$$

$$R_2(5\heartsuit) \subseteq v(\psi) \quad \text{and} \quad R_2(5\heartsuit) \neq \{5\heartsuit\}$$

$$\Updownarrow$$

$$5\heartsuit \models \Box_2\psi$$

"At time t_2, Alice knows that the suit of 5♡ is heart".

$$5\heartsuit \models \psi \quad \text{and} \quad R_2(5\heartsuit) \neq \{5\heartsuit\}$$

$$\Updownarrow$$

$$5\heartsuit \models \rhd_2\psi$$

"The suit of 5♡ is heart and at time t_2 Alice is interested to know it".

$\psi' =$ "the suit of the card is spade"

$$v(\psi') = \{i\spadesuit \ : \ i \in \{1, \dots, 10\} \cup \{J, Q, K\}\}$$

$$R_2(K\spadesuit) = \{K\spadesuit\}$$

$$\Downarrow$$

$$K\spadesuit \not\models \rhd_2 \psi'$$

Since $K\spadesuit \models \psi'$

$\psi' =$ "the suit of the card is spade"

$$v(\psi') = \{i\spadesuit \ : \ i \in \{1, \ldots, 10\} \cup \{J, Q, K\}\}$$

$$R_2(K\spadesuit) = \{K\spadesuit\}$$

$$\Downarrow$$

$$K\spadesuit \not\models \rhd_2 \psi'$$

Since $K\spadesuit \models \psi'$

"At time t_2, Alice is not interested to know the suit of card $K\spadesuit$"

Time $t_3 > t_2$

$\{3\diamondsuit, 4\diamondsuit, 5\diamondsuit\}$ $\{7\diamondsuit, 9\diamondsuit\}$ $\{1\heartsuit, 3\heartsuit\}$ $\{8\heartsuit, 9\heartsuit, 10\heartsuit\}$ $\{4\spadesuit, 5\spadesuit, 6\spadesuit\}$ $\{1\clubsuit, 2\clubsuit, 4\clubsuit\}$ $\{8\spadesuit, 9\spadesuit\}$

Time $t_3 > t_2$

$\{3\diamondsuit, 4\diamondsuit, 5\diamondsuit\}$ $\{7\diamondsuit, 9\diamondsuit\}$ $\{1\heartsuit, 3\heartsuit\}$ $\{8\heartsuit, 9\heartsuit, 10\heartsuit\}$ $\{4\spadesuit, 5\spadesuit, 6\spadesuit\}$ $\{1\clubsuit, 2\clubsuit, 4\clubsuit\}$ $\{8\spadesuit, 9\spadesuit\}$

$\chi =$ "the number of the card is greater than or equal to 7"

$7\diamondsuit \models \Box_3\chi$, $9\spadesuit \models \rhd_3\chi$ and $4\heartsuit \not\models \rhd_3\chi$.

Refinement sequence of $\mathcal{D}$ assigned to $(\mathcal{D}, \{R_1, R_2, R_3\}, v)$

t_3 $\{3\diamondsuit, 4\diamondsuit, 5\diamondsuit\}$ $\{7\diamondsuit, 9\diamondsuit\}$ $\{1\heartsuit, 3\heartsuit\}$ $\{8\heartsuit, 9\heartsuit, 10\heartsuit\}$ $\{4\spadesuit, 5\spadesuit, 6\spadesuit\}$ $\{8\spadesuit, 9\spadesuit\}$ $\{1\clubsuit, 2\clubsuit, 4\clubsuit\}$ $\{9\clubsuit, 10\clubsuit\}$

t_2 $\{i\diamondsuit \mid i \in \{1, \ldots, 10\}\}$ $\{i\heartsuit \mid i \in \{1, \ldots, 10\}\}$ $\{i\spadesuit \mid i \in \{1, \ldots, 10\}\}$ $\{i\clubsuit \mid i \in \{1, \ldots, 10\}\}$

t_1 $\{ij \in \mathcal{D} \mid c(ij) = \textbf{red}\}$ $\{ij \in \mathcal{D} \mid c(ij) = \textbf{black}\}$

Conclusions and Future Works

*OPS*5 system ↗ sequences of refinement of orthopairs

*OPS*5 system ↘ epistemic logic

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$OPS5$ system ↗ sequences of refinement of orthopairs

$OPS5$ system ↘ epistemic logic

We intend

- to develop a modal system assigned to refinement sequences of coverings of a universe;
- to interpret the operations between orthopairs in $OPS5$ seen as an epistemic logic.

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Thanks for your attention!