Hereditarily finite list superstructures and list structures

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Hereditarily finite and list superstructures
• Barwise: Admissible sets.
• Ershov: $\Sigma$—definability in hereditarily finite superstructures.
• Goncharov and Sviridenko: lists over the elements of a given abstract data type.
Hereditarily finite superstructure

Let $\mathcal{M}$ be a model of $\sigma$

- $HF_0(M) = M$
- $HF_{n+1}(M) = HF_n(M) \cup \mathcal{P}_\omega(HF_n(M))$
- $HF(M) = \bigcup_{n<\omega} HF_n(M)$

Then HF superstructure $\mathbb{HF}(\mathcal{M}) = \langle M, HF(M), \sigma \cup \{\emptyset, \in^2, U^1\} \rangle$
Hereditarily finite list superstructure

Let $\mathcal{M}$ be a model of $\sigma$

- elements of $S^0(M)$ are finite lists of $M$,
- elements of $S^{n+1}(M)$ are finite lists of $S^n(M) \cup M$.
- $S(M) = \bigcup_{n \in \omega} S^n(M)$

Then $HW(\mathcal{M}) = \langle M, S(M), \sigma \cup \{\text{head, tail, cons, nil, } \in\rangle \rangle$

- $\text{head}(\langle x_1, x_2, \ldots, x_n \rangle) = x_n$, $\text{head}(\text{nil}) = \text{nil}$
- $\text{tail}(\langle x_1, x_2, \ldots, x_{n+1} \rangle) = (\langle x_1, x_2, \ldots, x_n \rangle)$, $\text{tail}(\langle y \rangle) = \text{tail}(\text{nil}) = \text{nil}$
- $\text{cons}(\langle x_1, x_2, \ldots, x_n \rangle, y) = (\langle x_1, x_2, \ldots, x_n, y \rangle)$,
- $y \in \langle x_1, x_2, \ldots, x_n \rangle \iff y = x_i$, for some $1 \leq i \leq n$.
- $\langle y_1, y_2, \ldots, y_m \rangle \subseteq \langle x_1, x_2, \ldots, x_n \rangle \iff m \leq n$ and $y_i = x_i$, for all $1 \leq i \leq m$. 

Ershov

\( \mathcal{A} = \langle A; P_0^{n_0}, \ldots, P_k^{n_k} \rangle \) is \( \Sigma \)-definable in \( \mathbb{A} \), if there are \( \Sigma \)-formulas (with parameters in \( \mathbb{A} \)) \( S(x) \), \( E^+(x, y) \), \( E^-(x, y) \), \( \Psi_i^+(x_1, \ldots, x_{n_i}) \), \( \Psi_i^-(x_1, \ldots, x_{n_i}) \), \( i = 1, \ldots, k \), such that

1. \( S^* = \{ x \in \mathbb{A} | \mathbb{A} \models S(x) \} \neq \emptyset \),
2. \( E^+(x, y) \) defines congruence \( \eta \) on \( \mathcal{A}^* = \langle S^*; P_1^*, \ldots, P_k^* \rangle \),
   \( (P_i^* = \{ < x_1, \ldots, x_{n_i} > | \mathbb{A} \models \Psi_i^+(x_1, \ldots, x_{n_i}) \}) \)
3. sets, defined by \( E^+ \) и \( E^- \), have empty intersection and their union is \( (S^*)^2 \),
4. sets, defined by \( \Psi_i^+ \) и \( \Psi_i^- \), have empty intersection and their union is \( (S^*)^n \),
5. \( \mathcal{A}^*/\eta \cong \mathcal{A} \).
Morozov

$A$ is $\Sigma$-definable in $B$ if $i$ is an embedding from the definition $A \subseteq \Sigma B$, there is a $\Sigma$-function $\sigma$ over $B$ such that $\text{dom}(\sigma) = i[A]$ and $\sigma(i(x)) = \{i(y) \mid y \in x\}$ holds for all $x \in A$.

Basic property:

For each $\Sigma$-subset $S$ of $A^n \{\langle i(x_1), \ldots, i(x_n) \rangle \mid \langle x_1, \ldots, x_n \rangle \in S\}$ is a $\Sigma$-subset of $B$.

Puzarenko

$A$ is $\Sigma$-definable in $B$ if $i[\Sigma(A^2)] \subseteq \Sigma(B^2)$. 
Morozov

\( \mathcal{A} \) is \( \Sigma \)-definable in \( \mathcal{B} \) if \( i \) is an embedding from the definition \( \mathcal{A} \sqsubseteq_{\Sigma} \mathcal{B} \), there is a \( \Sigma \)-function \( \sigma \) over \( \mathcal{B} \) such that \( \text{dom}(\sigma) = i[\mathcal{A}] \) and \( \sigma(i(x)) = \{ i(y) \mid y \in x \} \) holds for all \( x \in \mathcal{A} \).

Basic property:

For each \( \Sigma \)-subset \( S \) of \( \mathcal{A}^n \) \( \{ \langle i(x_1), \ldots, i(x_n) \rangle \mid \langle x_1, \ldots, x_n \rangle \in S \} \) is a \( \Sigma \)-subset of \( \mathcal{B} \).

Puzarenko

\( \mathcal{A} \) is \( \Sigma \)-definable in \( \mathcal{B} \) if \( i[\Sigma(\mathcal{A}^2)] \subseteq \Sigma(\mathcal{B}^2) \).
Theorem

$HF(M)$ is $\Sigma$–definable in $HW(M)$ and $HW(M)$ is $\Sigma$–definable in $HF(M)$.

- The isomorphism $A^*/\eta \cong A$ in both cases is identical on $M$. 

Σ–definability of superstructures
Corollary

- For every $\Sigma$–formula $\Phi(x)$, $x \in M$ there is some $\Psi(y)$ such that $\text{HF}(M) \models \Phi(x) \iff \text{HW}(M) \models \Psi(y)$.
- For every $\Sigma$–formula $\Psi(y)$, $y \in M$ there is some $\Phi(x)$ such that $\text{HW}(M) \models \Psi(y) \iff \text{HF}(M) \models \Phi(x)$.

Corollary

$X \subseteq M$ is $\Sigma$–definable in $\text{HF}(M)$ iff $X$ is $\Sigma$–definable in $\text{HW}(M)$.
List structures (feat. N. Bazhenov)
Motivation

• Moore and Russell 1981: axiomatic theory of lists.
• Goncharov 1983: matrices as lists of lists.
• Goncharov 1986: lists over the elements of a given abstract data type.

Goncharov (1986) proved that the theory of lists over models of a decidable theory is decidable.
List structure

Two sorts of variables: atom and list.

For a finite signature $\sigma$, $L_\sigma$ is a two-sorted first-order language with:

- $nil$: list,
- $cons$: list $\times$ atom $\to$ list.

**List structure over** $\mathcal{M}$ is the structure $LS(\mathcal{M})$ of the language $L_\sigma$ such that

- atom $= |\mathcal{M}|$;
- list $= |\mathcal{M}|^{<\omega}$;
- $nil = \Lambda$;
- $cons(\Lambda, a) = \langle a \rangle$ and $cons(\langle a_0, \ldots a_n \rangle, b) = \langle a_0, \ldots a_n, b \rangle$. 
We add to the language $L_\sigma$ new symbols:

- **head**: list $\rightarrow$ atom $\cup$ list,
- **tail**: list $\rightarrow$ list,
- **$\sqsubseteq$** list $\times$ list,
- **$\in$** atom $\times$ list.

**Enriched list structure** over $M$ ($ELS(M)$):

- $head(\Lambda) = \Lambda$ and $head(\langle a_1, \ldots, a_n \rangle) = a_n$,
- $tail(\Lambda) = tail(\langle a \rangle) = \Lambda$ and
  $tail(\langle a_1, \ldots, a_n, a_{n+1} \rangle) = \langle a_1, \ldots, a_n \rangle$,
- $x \sqsubseteq y$ if a list $x$ is an initial segment of a list $y$,
- $a \in x$ if $a$ is an element from a list $x$. 
Let $\mathcal{M}$ be an $L$-structure, $\alpha$ computable ordinal, and $\Psi = \{\psi_n(\bar{x}_n)\}_{n \in \omega}$ is a uniformly computable sequence of computable infinitary formulas in the language $L \cup \{\text{nil}, \text{cons}\}$.

The $\Psi$-$S(\mathcal{M})$ is a two-sorted structure in the language $L^\Psi := L \cup \{R_{\psi_n}\}_{n \in \omega}$, where $R_{\psi_n}$ are new symbols, such that:

- any symbol from $L$ is treated as applying only to atoms;
- for each $n \in \omega$, $R_{\psi_n}$ is interpreted as $\psi_n[LS(\mathcal{M})]$. 
Problem

Suppose that $\mathcal{M}$ is a countable structure, and $S(\mathcal{M})$ is a list-extended structure over $\mathcal{M}$ (say, $LS(\mathcal{M})$ or $ELS(\mathcal{M})$).

- If the first-order theory of $\mathcal{M}$ is decidable, then is it true that the theory of $S(\mathcal{M})$ is also decidable?

Proposition (Goncharov 1986)

If the first-order theory of $\mathcal{M}$ is decidable, then the theory of $LS(\mathcal{M})$ is also decidable.
Decidability results

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Decidability results

Theorem

Let $\mathcal{M} = (\omega; +)$, i.e. $\mathcal{M}$. Then the theory of $ELS(\mathcal{M})$ is undecidable.

Corollary

The list-extended structure $\{\text{head}, \text{tail}, \sqsubseteq\} - S(\omega, +)$ has no decidable copies.

Corollary

The theory of $\{\text{head}, \text{tail}, \sqsubseteq\} - S(\omega, +)$ is computably isomorphic to $\emptyset(\omega)$. 
Decidability results

Theorem

Let $A$ be non-empty, at most countable set. Then the first-order theory of $ELS^2(A)$ is computably isomorphic to the first-order arithmetic.

Corollary

Let $\mathcal{M}$ be an $L$-structure such that its atomic diagram $D(\mathcal{M})$ is arithmetical, i.e. $D(\mathcal{M}) \leq_T \emptyset^{(n)}$ for some $n \in \omega$. Then the theory of $ELS^2(\mathcal{M})$ is computably isomorphic to the first-order arithmetic.
Thank you!