Hereditarily finite list superstructures and list structures

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Hereditarily finite and list superstructures

Introduction

- Barwise: Admissible sets.
- Ershov: $\Sigma-$definability in hereditarily finite superstructures.
- Goncharov and Sviridenko: lists over the elements of a given abstract data type.

Hereditarily finite superstructure

Let $\mathcal{M}$ be a model of σ

- $HF_0(M) = M$
- $HF_{n+1}(M) = HF_n(M) \cup \mathcal{P}_\omega(HF_n(M))$
- $HF(M) = \bigcup_{n<\omega} HF_n(M)$

Then HF superstructure $\mathbb{HF}(\mathcal{M}) = \langle M, HF(M), \sigma \cup \{\varnothing, \in^2, U^1\}\rangle$

Hereditarily finite list superstructure

Let $\mathcal{M}$ be a model of σ

- elements of $S^0(M)$ are finite lists of M,
- elements of $S^{n+1}(M)$ are finite lists of $S^n(M) \cup M$.
- $S(M) = \bigcup_{n \in \omega} S^n(M)$

Then $HW(\mathcal{M}) = \langle M, S(M), \sigma \cup \{head, tail, cons, nil, \in\} \rangle$

- $head(< x_1, x_2, \ldots, x_n >) = x_n$,
  $head(nil) = nil$
- $tail(< x_1, x_2, \ldots, x_{n+1} >) = (< x_1, x_2, \ldots, x_n >)$,
  $tail(< y >) = tail(nil) = nil$
- $cons(< x_1, x_2, \ldots, x_n >, y) = (< x_1, x_2, \ldots, x_n, y >)$,
- $y \in < x_1, x_2, \ldots, x_n > \iff y = x_i$, for some $1 \leq i \leq n$.
- $< y_1, y_2, \ldots, y_m > \sqsubseteq < x_1, x_2, \ldots, x_n > \iff m \leq n$ and $y_i = x_i$, for all $1 \leq i \leq m$.

$\Sigma-$definability in admissible sets

Ershov

$\mathcal{A} = \langle A; P_0^{n_0}, \ldots, P_k^{n_k} \rangle$ is $\Sigma-$definable in $\mathbb{A}$, if there are $\Sigma-$formulas (with parameters in $\mathbb{A}$) $S(x)$, $E^+(x, y)$, $E^-(x, y)$, $\Psi_i^+(x_1, \ldots, x_{n_i})$, $\Psi_i^-(x_1, \ldots, x_{n_i})$, $i = 1, \ldots, k$, such that

1. $S^* = \{x \in \mathbb{A} | \mathbb{A} \models S(x)\} \neq \varnothing$,
2. $E^+(x, y)$ defines congruence η on $\mathcal{A}^* = \langle S^*; P_1^*, \ldots, P_k^* \rangle$, $(P_i^* = \{< x_1, \ldots, x_{n_i} > | \mathbb{A} \models \Psi_i^+(x_1, \ldots, x_{n_i})\})$
3. sets, defined by E^+ и E^-, have empty intersection and their union is $(S^*)^2$,
4. sets, defined by Ψ_i^+ и Ψ_i^+, have empty intersection and their union is $(S^*)^n$,
5. $\mathcal{A}^*/\eta \cong \mathcal{A}$.

Σ-definability in admissible sets

Morozov

$\mathbb{A}$ is Σ-definable in $\mathbb{B}$ if i is an embedding from the definition $\mathbb{A} \sqsubseteq_{\Sigma} \mathbb{B}$, there is a Σ-function σ over $\mathbb{B}$ such that $dom(\sigma) = i[\mathbb{A}]$ and $\sigma(i(x)) = \{i(y) | y \in x\}$ holds for all $x \in \mathbb{A}$.

Basic property:

For each Σ-subset S of $\mathbb{A}^n$ $\{\langle i(x_1), ..., i(x_n)\rangle | \langle x_1, ..., x_n\rangle \in S\}$ is a Σ-subset of $\mathbb{B}$.

Puzarenko

$\mathbb{A}$ is Σ-definable in $\mathbb{B}$ if $i[\Sigma(\mathbb{A}^2)] \subseteq \Sigma(\mathbb{B}^2)$.

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$\Sigma-$definability of superstructures

Theorem

$\mathbb{HF}(\mathcal{M})$ is $\Sigma-$definable in $HW(\mathcal{M})$ and $HW(\mathcal{M})$ is $\Sigma-$definable in $\mathbb{HF}(\mathcal{M})$.

- The isomorphism $\mathcal{A}^*/\eta \cong \mathcal{A}$ in both cases is identical on $\mathcal{M}$.

Corollary

- For every $\Sigma-$formula $\Phi(\overline{x}), x \in M$ there is some $\Psi(\overline{y})$ such that $\mathbb{HF}(\mathcal{M}) \models \Phi(\overline{x}) \leftrightarrow HW(\mathcal{M}) \models \Psi(\overline{x})$.
- For every $\Sigma-$formula $\Psi(\overline{y}), y \in M$ there is some $\Phi(\overline{x})$ such that $HW(\mathcal{M}) \models \Psi(\overline{y}) \leftrightarrow \mathbb{HF}(\mathcal{M}) \models \Phi(\overline{x})$.

Corollary

$X \subseteq M$ is $\Sigma-$definable in $\mathbb{HF}(\mathcal{M})$ iff X is $\Sigma-$definable in $HW(\mathcal{M})$

List structures (feat. N. Bazhenov)

Motivation

- Moore and Russell 1981: axiomatic theory of lists.
- Goncharov 1983: matrices as lists of lists.
- Goncharov 1986: lists over the elements of a given abstract data type.

Goncharov (1986) proved that the theory of lists over models of a decidable theory is decidable.

List structure

Two sorts of variables: `atom` and `list`.

For a finite signature σ, L_σ is a two-sorted first-order language with:

- *nil*: $\texttt{list}$,
- *cons*: $\texttt{list} \times \texttt{atom} \to \texttt{list}$.

List structure over $\mathcal{M}$ is the structure $LS(\mathcal{M})$ of the language L_σ such that

- $\texttt{atom} = |\mathcal{M}|$;
- $\texttt{list} = |\mathcal{M}|^{<\omega}$;
- $nil = \Lambda$;
- $cons(\Lambda, a) = \langle a \rangle$ and $cons(\langle a_0, \ldots a_n \rangle, b) = \langle a_0, \ldots a_n, b \rangle$.

Enriched list structure

We add to the language L_σ new symbols:

- *head*: $\texttt{list} \to \texttt{atom} \cup \texttt{list}$,
- *tail*: $\texttt{list} \to \texttt{list}$,
- $\sqsubseteq \subseteq \texttt{list} \times \texttt{list}$,
- $\in \subseteq \texttt{atom} \times \texttt{list}$.

Enriched list structure over $\mathcal{M}$ ($ELS(\mathcal{M})$):

- $head(\Lambda) = \Lambda$ and $head(\langle a_1, \ldots, a_n \rangle) = a_n$,
- $tail(\Lambda) = tail(\langle a \rangle) = \Lambda$ and $tail(\langle a_1, \ldots, a_n, a_{n+1} \rangle) = \langle a_1, \ldots, a_n \rangle$,
- $x \sqsubseteq y$ if a list x is an initial segment of a list y,
- $a \in x$ if a is an element from a list x.

Extended list structure

Let $\mathcal{M}$ be an L-structure, α computable ordinal, and $\Psi = \{\psi_n(\bar{x}_n)\}_{n\in\omega}$ is a uniformly computable sequence of computable infinitary formulas in the language $L \cup \{nil, cons\}$.

The Ψ-$S(\mathcal{M})$ is a two-sorted structure in the language $L^\Psi := L \cup \{R_{\psi_n}\}_{n\in\omega}$, where R_{ψ_n} are new symbols, such that:

- any symbol from L is treated as applying only to atoms;
- for each $n \in \omega$, R_{ψ_n} is interpreted as $\psi_n[LS(\mathcal{M})]$.

Decidability results

Problem

Suppose that $\mathcal{M}$ is a countable structure, and $S(\mathcal{M})$ is a list-extended structure over $\mathcal{M}$ (say, $LS(\mathcal{M})$ or $ELS(\mathcal{M})$).

- *If the first-order theory of $\mathcal{M}$ is decidable, then is it true that the theory of $S(\mathcal{M})$ is also decidable?*

Proposition (Goncharov 1986)

If the first-order theory of $\mathcal{M}$ is decidable, then the theory of $LS(\mathcal{M})$ is also decidable.

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Decidability results

Theorem

Let $\mathcal{M} = (\omega; +)$, *i.e.* $\mathcal{M}$. *Then the theory of* $ELS(\mathcal{M})$ *is undecidable.*

Corollary

The list-extended structure $\{head, tail, \sqsubseteq\}$-$S(\omega, +)$ *has no decidable copies.*

Corollary

The theory of $\{head, tail, \sqsubseteq\}$-$S(\omega, +)$ *is computably isomorphic to* $\emptyset^{(\omega)}$.

Decidability results

Theorem

Let A be non-empty, at most countable set. Then the first-order theory of $ELS^2(A)$ is computably isomorphic to the first-order arithmetic.

Corollary

Let $\mathcal{M}$ be an L-structure such that its atomic diagram $D(\mathcal{M})$ is arithmetical, i.e. $D(\mathcal{M}) \leq_T \emptyset^{(n)}$ for some $n \in \omega$. Then the theory of $ELS^2(\mathcal{M})$ is computably isomorphic to the first-order arithmetic.

Thank you!