Logics of variable inclusion and Płonka sums of matrices

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July 20, 2018

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- Recently, Płonka sums have found some applications in the realm of paraconsistent logics as well.
- We develop a general theory of Płonka sums which applies to all infinitary Universal Horn Theories without equality.
- However, for the sake of simplicity, we confine our attention to the special case of propositional logics, i.e. substitution-invariant consequence relations over the set of terms (in an infinite set of variables) of a given algebraic language.

Definition

The left variable inclusion companion of a logic $\vdash$ is that relation $\vdash^l$ defined as follows for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash^l \varphi \iff \text{ there is } \Gamma' \subseteq \Gamma \text{ s.t. } \mathsf{Var}(\Gamma') \subseteq \mathsf{Var}(\varphi) \text{ and } \Gamma' \vdash \varphi.$$

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- The relation $\vdash^{l}$ is also a logic.
- The left variable inclusion companion of Classical Logic is the so-called Paraconsistent Weak Kleene Logic $\boldsymbol{PWK}$.
- In order to make explicit the relation between left variable inclusion companions and Płonka sums, we need an additional definition.

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- In this case, $\mathcal{P}(\boldsymbol{A})_{i \in I}$ is the algebra with universe $\bigcup_{i \in I} A_i$ such that for every $a_1 \in A_{m_1}, \ldots, a_n \in A_{m_n}$,

$$f^{\mathcal{P}(\boldsymbol{A})_{i \in I}}(a_1, \ldots, a_n) \coloneqq f^{\boldsymbol{A}_j}(f_{m_1 j}(a_1), \ldots, f_{m_n j}(a_n)),$$

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- The Płonka sum of $\mathbb{X}$ is the matrix

$$\mathcal{P}(\mathbb{X}) \coloneqq \langle \mathcal{P}(\boldsymbol{A})_{i \in I}, \bigcup_{i \in I} F_i \rangle.$$

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- This observation provides a semantic description of left variable inclusion companions as logics of Płonka sums of matrices.
- It is natural to wonder whether we can produce an axiomatic description as well.

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- In Hilbert algebras $x \cdot y \coloneqq (y \to y) \to x$.

- Let $\mathbb{H}$ be a Hilbert-style calculus with finite rules, which determines a logic $\vdash$ with a partition function $\cdot$.

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Theorem (Axiomatization)

Let $\vdash$ be a finitary logic with partition function $\cdot$ axiomatized by a Hilbert calculus $\mathbb{H}$. Then $\mathbb{LH}$ is a complete Hilbert calculus for $\vdash^l$.

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$$\varphi, \varphi \to \psi \rhd \psi \wedge (\psi \vee (\varphi \wedge (\varphi \vee (\varphi \to \psi))))$$
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for all $\chi(\delta, \vec{z}) \dashv\vdash \chi(\epsilon, \vec{z})$ in the def. of partition function.

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2. There exists a direct system of matrices $\mathbb{X} \subseteq \mathrm{Mod}^{\mathrm{Su}}(\vdash)$ indexed by a semilattice I such that $\langle \boldsymbol{A}, F \rangle \cong \mathcal{P}(\mathbb{X})$ and for every $n, i \in I$ such that $\langle \boldsymbol{A}_n, F_n \rangle$ is trivial and $n < i$, there exists $j \in I$ s.t. $n \leqslant j, i \nleqslant j$ and $\boldsymbol{A}_j$ is non-trivial.

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- As a consequence, all models in $\mathrm{Mod}^{\mathrm{Su}}(\vdash^{l})$ can be represented as Płonka sums of models in $\mathrm{Mod}^{\mathrm{Su}}(\vdash)$.

- From a semantic point of view, it is natural to look for a description of $\mathrm{Mod}^{Su}(\vdash^l)$ in terms of $\mathrm{Mod}^{Su}(\vdash)$.

Theorem

Let $\vdash$ be an equivalential and finitary logic with a partition function. TFAE:

1. $\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}^{\mathrm{Su}}(\vdash^l)$.
2. There exists a direct system of matrices $\mathbb{X} \subseteq \mathrm{Mod}^{\mathrm{Su}}(\vdash)$ indexed by a semilattice I such that $\langle \boldsymbol{A}, F\rangle \cong \mathcal{P}(\mathbb{X})$ and for every $n, i \in I$ such that $\langle \boldsymbol{A}_n, F_n\rangle$ is trivial and $n < i$, there exists $j \in I$ s.t. $n \leqslant j, i \nleqslant j$ and $\boldsymbol{A}_j$ is non-trivial.

- As a consequence, all models in $\mathrm{Mod}^{\mathrm{Su}}(\vdash^l)$ can be represented as Płonka sums of models in $\mathrm{Mod}^{\mathrm{Su}}(\vdash)$.
- Is any simplification possible, on general grounds?

Definition

A logic $\vdash$ has a set of anti-theorem if there is a set of formulas Σ such that $\sigma[\Sigma] \vdash \varphi$ for every substitution σ and formula φ.

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A logic $\vdash$ has a set of inconsistency terms iff nontrivial models of $\vdash$ lack trivial submodels.

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Lemma

A logic $\vdash$ has a set of inconsistency terms iff nontrivial models of $\vdash$ lack trivial submodels.

Corollary

Let $\vdash$ be an equivalential and finitary logic with a partition function and inconsistency terms. TFAE:

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A logic $\vdash$ has a set of inconsistency terms iff nontrivial models of $\vdash$ lack trivial submodels.

Corollary

Let $\vdash$ be an equivalential and finitary logic with a partition function and inconsistency terms. TFAE:

1. $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{\mathsf{Su}}(\vdash^{l})$.
2. There exists a direct system of matrices $\mathbb{X} \subseteq \mathsf{Mod}^{\mathsf{Su}}(\vdash)$ with at most one trivial component such that $\langle \boldsymbol{A}, F \rangle \cong \mathcal{P}(\mathbb{X})$.

...thank you for coming!