Relational semantics, ordered algebras, and quantifiers for deductive systems

Tommaso Moraschini
joint work with Ramon Jansana

Institute of Computer Science of the Czech Academy of Sciences

July 23, 2018

Contents

1. What is a frame? (for an arbitrary algebraic language)

2. What does it mean that a logic has a local relational semantics?

3. Why do most logics have a semantics of ordered algebras?

4. Are there logic-based dualities/completions for ordered algebras?

Definition

1. An order type for an algebraic language $\mathscr{L}$ is an assignment to every symbol $f \in \mathscr{L}$ of a choice of which arguments of f will be treated as increasing and which ones as decreasing.

Definition

1. An order type for an algebraic language $\mathscr{L}$ is an assignment to every symbol $f \in \mathscr{L}$ of a choice of which arguments of f will be treated as increasing and which ones as decreasing.
2. An ordered language is an algebraic language equipped with an order type.

Definition

1. An order type for an algebraic language $\mathscr{L}$ is an assignment to every symbol $f \in \mathscr{L}$ of a choice of which arguments of f will be treated as increasing and which ones as decreasing.
2. An ordered language is an algebraic language equipped with an order type.
3. Let $\mathscr{L}$ be an ordered language. An $\mathscr{L}$-algebra is a pair $\langle \boldsymbol{A}, \leqslant \rangle$ where $\boldsymbol{A}$ is an algebra, $\leqslant$ a partial order on A, and if $f = f(\vec{x}, \vec{y})$, then $f^{\boldsymbol{A}}$ is incr. on $\vec{x}$ and decr. on $\vec{y}$ w.r.t. $\leqslant$.

Definition

1. An order type for an algebraic language $\mathscr{L}$ is an assignment to every symbol $f \in \mathscr{L}$ of a choice of which arguments of f will be treated as increasing and which ones as decreasing.
2. An ordered language is an algebraic language equipped with an order type.
3. Let $\mathscr{L}$ be an ordered language. An $\mathscr{L}$-algebra is a pair $\langle \boldsymbol{A}, \leqslant \rangle$ where $\boldsymbol{A}$ is an algebra, $\leqslant$ a partial order on A, and if $f = f(\vec{x}, \vec{y})$, then $f^{\boldsymbol{A}}$ is incr. on $\vec{x}$ and decr. on $\vec{y}$ w.r.t. $\leqslant$.

Examples:

Definition

1. An order type for an algebraic language $\mathscr{L}$ is an assignment to every symbol $f \in \mathscr{L}$ of a choice of which arguments of f will be treated as increasing and which ones as decreasing.
2. An ordered language is an algebraic language equipped with an order type.
3. Let $\mathscr{L}$ be an ordered language. An $\mathscr{L}$-algebra is a pair $\langle \boldsymbol{A}, \leqslant \rangle$ where $\boldsymbol{A}$ is an algebra, $\leqslant$ a partial order on A, and if $f = f(\vec{x}, \vec{y})$, then $f^{\boldsymbol{A}}$ is incr. on $\vec{x}$ and decr. on $\vec{y}$ w.r.t. $\leqslant$.

Examples:

- Heyting algebras, modal algebras and residuated lattices equipped with the lattice order can be seen as $\mathscr{L}$-algebras.

Definition

A polarity is a triple $\langle W, J, R\rangle$ such that W and J are non-empty sets and $R \subseteq W \times J$.

Definition

A polarity is a triple $\langle W, J, R\rangle$ such that W and J are non-empty sets and $R \subseteq W \times J$.

- Every polarity $\langle W, J, R\rangle$ induces a Galois connection

$$(\cdot)^{\triangleright} : \mathcal{P}(W) \longleftrightarrow \mathcal{P}(J) : (\cdot)^{\triangleleft}$$

Definition

A polarity is a triple $\langle W, J, R\rangle$ such that W and J are non-empty sets and $R \subseteq W \times J$.

- Every polarity $\langle W, J, R\rangle$ induces a Galois connection

$$(\cdot)^{\rhd} : \mathcal{P}(W) \longleftrightarrow \mathcal{P}(J) : (\cdot)^{\lhd}$$

by setting for $A \subseteq W$ and $B \subseteq J$

$$A^{\rhd} := \{j \in J : \langle w, j\rangle \in R \text{ for all } w \in A\}$$
$$B^{\lhd} := \{w \in W : \langle w, j\rangle \in R \text{ for all } j \in B\}.$$

Definition

A polarity is a triple $\langle W, J, R\rangle$ such that W and J are non-empty sets and $R \subseteq W \times J$.

- Every polarity $\langle W, J, R\rangle$ induces a Galois connection

$$(\cdot)^{\triangleright} \colon \mathcal{P}(W) \longleftrightarrow \mathcal{P}(J) \colon (\cdot)^{\triangleleft}$$

by setting for $A \subseteq W$ and $B \subseteq J$

$$A^{\triangleright} := \{j \in J : \langle w, j\rangle \in R \text{ for all } w \in A\}$$
$$B^{\triangleleft} := \{w \in W : \langle w, j\rangle \in R \text{ for all } j \in B\}.$$

- Indeed, we have that $B \subseteq A^{\triangleright} \iff A \subseteq B^{\triangleleft}$.

Definition

A polarity is a triple $\langle W, J, R\rangle$ such that W and J are non-empty sets and $R \subseteq W \times J$.

- Every polarity $\langle W, J, R\rangle$ induces a Galois connection

$$(\cdot)^{\rhd} : \mathcal{P}(W) \longleftrightarrow \mathcal{P}(J) : (\cdot)^{\lhd}$$

by setting for $A \subseteq W$ and $B \subseteq J$

$$A^{\rhd} := \{j \in J : \langle w, j\rangle \in R \text{ for all } w \in A\}$$
$$B^{\lhd} := \{w \in W : \langle w, j\rangle \in R \text{ for all } j \in B\}.$$

- Indeed, we have that $B \subseteq A^{\rhd} \iff A \subseteq B^{\lhd}$.
- Then $(\cdot)^{\rhd\lhd} : \mathcal{P}(W) \to \mathcal{P}(W)$ is a closure operator on W.

Definition

A polarity is a triple $\langle W, J, R\rangle$ such that W and J are non-empty sets and $R \subseteq W \times J$.

- Every polarity $\langle W, J, R\rangle$ induces a Galois connection

$$(\cdot)^{\triangleright} \colon \mathcal{P}(W) \longleftrightarrow \mathcal{P}(J) \colon (\cdot)^{\triangleleft}$$

by setting for $A \subseteq W$ and $B \subseteq J$

$$A^{\triangleright} := \{j \in J : \langle w, j\rangle \in R \text{ for all } w \in A\}$$
$$B^{\triangleleft} := \{w \in W : \langle w, j\rangle \in R \text{ for all } j \in B\}.$$

- Indeed, we have that $B \subseteq A^{\triangleright} \iff A \subseteq B^{\triangleleft}$.
- Then $(\cdot)^{\triangleright\triangleleft} \colon \mathcal{P}(W) \to \mathcal{P}(W)$ is a closure operator on W. We denote its lattice of closed sets by $\mathcal{G}(W, J, R)$.

1. A labeling map for an algebraic language $\mathscr{L}$ is a function $\beta\colon \mathscr{L} \to \{\Box, \Diamond\}$.

1. A labeling map for an algebraic language $\mathscr{L}$ is a function $\beta\colon \mathscr{L} \to \{\Box, \Diamond\}$.
2. A labeled language is an algebraic language $\mathscr{L}$ equipped with a labeling map β. Sometimes we write $\mathscr{L}^{\beta}$.

1. A labeling map for an algebraic language $\mathscr{L}$ is a function $\beta\colon \mathscr{L} \to \{\Box, \Diamond\}$.
2. A labeled language is an algebraic language $\mathscr{L}$ equipped with a labeling map β. Sometimes we write $\mathscr{L}^\beta$.

Definition

Let $\mathscr{L}$ be a labeled ordered language.

1. A labeling map for an algebraic language $\mathscr{L}$ is a function $\beta \colon \mathscr{L} \to \{\Box, \Diamond\}$.
2. A labeled language is an algebraic language $\mathscr{L}$ equipped with a labeling map β. Sometimes we write $\mathscr{L}^{\beta}$.

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-preframe is a structure

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

1. A labeling map for an algebraic language $\mathscr{L}$ is a function $\beta\colon \mathscr{L} \to \{\Box, \Diamond\}$.
2. A labeled language is an algebraic language $\mathscr{L}$ equipped with a labeling map β. Sometimes we write $\mathscr{L}^{\beta}$.

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-preframe is a structure

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

where $\langle W, J, R\rangle$ is a polarity such that $\leqslant_W$ and $\leqslant_J$ are partial orders,

1. A labeling map for an algebraic language $\mathscr{L}$ is a function $\beta\colon \mathscr{L} \to \{\Box, \Diamond\}$.
2. A labeled language is an algebraic language $\mathscr{L}$ equipped with a labeling map β. Sometimes we write $\mathscr{L}^{\beta}$.

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-preframe is a structure

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

where $\langle W, J, R\rangle$ is a polarity such that $\leqslant_W$ and $\leqslant_J$ are partial orders, and for every operation symbol $f \in \mathscr{L}$ such that $f = f(x_1, \dots, x_m; y_1, \dots, y_n)$ we have:

1. A labeling map for an algebraic language $\mathscr{L}$ is a function $\beta\colon \mathscr{L} \to \{\Box, \Diamond\}$.
2. A labeled language is an algebraic language $\mathscr{L}$ equipped with a labeling map β. Sometimes we write $\mathscr{L}^\beta$.

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-preframe is a structure

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

where $\langle W, J, R\rangle$ is a polarity such that $\leqslant_W$ and $\leqslant_J$ are partial orders, and for every operation symbol $f \in \mathscr{L}$ such that $f = f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ we have:

$$\text{if } \beta(f) = \Diamond, \text{ then } T_f \subseteq W^m \times J^n \times W$$
$$\text{if } \beta(f) = \Box, \text{ then } T_f \subseteq J^m \times W^n \times J.$$

Definition

Let $\mathscr{L}$ be a labeled ordered language.

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-frame

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

is an $\mathscr{L}$-preframe such that for all connectives $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ s.t. $\beta(f) = \Diamond$:

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-frame

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

is an $\mathscr{L}$-preframe such that for all connectives $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ s.t. $\beta(f) = \Diamond$:

(a) For every $\vec{w}_1, \vec{w}_2 \in W^m$, $\vec{j}_1, \vec{j}_2 \in J^n$, and $u_1, u_2 \in W$ such that $\vec{w}_2 \leqslant_W \vec{w}_1$, $\vec{j}_2 \leqslant_J \vec{j}_1$ and $u_1 \leqslant_W u_2$,

$$\text{if } \langle \vec{w}_1, \vec{j}_1, u_1\rangle \in T_f, \text{ then } \langle \vec{w}_2, \vec{j}_2, u_2\rangle \in T_f.$$

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-frame

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

is an $\mathscr{L}$-preframe such that for all connectives $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ s.t. $\beta(f) = \diamond$:

(a) For every $\vec{w}_1, \vec{w}_2 \in W^m$, $\vec{j}_1, \vec{j}_2 \in J^n$, and $u_1, u_2 \in W$ such that $\vec{w}_2 \leqslant_W \vec{w}_1$, $\vec{j}_2 \leqslant_J \vec{j}_1$ and $u_1 \leqslant_W u_2$,

$$\text{if } \langle \vec{w}_1, \vec{j}_1, u_1\rangle \in T_f, \text{ then } \langle \vec{w}_2, \vec{j}_2, u_2\rangle \in T_f.$$

(b) $T_f(\vec{w}, \vec{j})$ is a closed set of $(\cdot)^{\rhd\lhd}$ for all $\vec{w} \in W^m$ and $\vec{j} \in J^n$.

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-frame

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

is an $\mathscr{L}$-preframe such that for all connectives $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ s.t. $\beta(f) = \Diamond$:

(a) For every $\vec{w}_1, \vec{w}_2 \in W^m$, $\vec{j}_1, \vec{j}_2 \in J^n$, and $u_1, u_2 \in W$ such that $\vec{w}_2 \leqslant_W \vec{w}_1$, $\vec{j}_2 \leqslant_J \vec{j}_1$ and $u_1 \leqslant_W u_2$,

$$\text{if } \langle \vec{w}_1, \vec{j}_1, u_1\rangle \in T_f, \text{ then } \langle \vec{w}_2, \vec{j}_2, u_2\rangle \in T_f.$$

(b) $T_f(\vec{w}, \vec{j})$ is a closed set of $(\cdot)^{\rhd\lhd}$ for all $\vec{w} \in W^m$ and $\vec{j} \in J^n$.

Connectives f such that $\beta(f) = \Box$ need to satisfy a dual requirement.

Definition

Let $\mathscr{L}$ be a labeled ordered language. An $\mathscr{L}$-frame

$$\boldsymbol{F} = \langle W, J, R, \{T_f : f \in \mathscr{L}\}\rangle$$

is an $\mathscr{L}$-preframe such that for all connectives $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ s.t. $\beta(f) = \diamond$:

(a) For every $\vec{w}_1, \vec{w}_2 \in W^m$, $\vec{j}_1, \vec{j}_2 \in J^n$, and $u_1, u_2 \in W$ such that $\vec{w}_2 \leqslant_W \vec{w}_1$, $\vec{j}_2 \leqslant_J \vec{j}_1$ and $u_1 \leqslant_W u_2$,

$$\text{if } \langle \vec{w}_1, \vec{j}_1, u_1\rangle \in T_f, \text{ then } \langle \vec{w}_2, \vec{j}_2, u_2\rangle \in T_f.$$

(b) $T_f(\vec{w}, \vec{j})$ is a closed set of $(\cdot)^{\rhd\lhd}$ for all $\vec{w} \in W^m$ and $\vec{j} \in J^n$.

Connectives f such that $\beta(f) = \Box$ need to satisfy a dual requirement.

We refer to W and J as to the sets of worlds and co-worlds of $\boldsymbol{F}$ respectively.

- A valuation in a $\mathscr{L}$-frame $\boldsymbol{F}$ is a map $v\colon Var \to \mathcal{G}(W, J, R)$.

- A valuation in a $\mathscr{L}$-frame $\boldsymbol{F}$ is a map $v\colon Var \to \mathcal{G}(W, J, R)$.
- We want to define two relations of satisfaction and co-satisfaction of formulas under v, respectively at worlds $w \in W$ and co-worlds $j \in J$, in symbols

$$w, v \Vdash \varphi \text{ and } j, v \succ \varphi.$$

- A valuation in a $\mathscr{L}$-frame $\boldsymbol{F}$ is a map $v\colon Var \to \mathcal{G}(W, J, R)$.
- We want to define two relations of satisfaction and co-satisfaction of formulas under v, respectively at worlds $w \in W$ and co-worlds $j \in J$, in symbols

$$w, v \Vdash \varphi \text{ and } j, v \succ \varphi.$$

- For every variable $x \in Var$, we set

$$w, v \Vdash x \iff w \in v(x)$$
$$j, v \succ x \iff j \in v(x)^{\triangleright}.$$

- A valuation in a $\mathscr{L}$-frame $\boldsymbol{F}$ is a map $v\colon Var \to \mathcal{G}(W, J, R)$.
- We want to define two relations of satisfaction and co-satisfaction of formulas under v, respectively at worlds $w \in W$ and co-worlds $j \in J$, in symbols

$$w, v \Vdash \varphi \text{ and } j, v \succ \varphi.$$

- For every variable $x \in Var$, we set

$$w, v \Vdash x \Longleftrightarrow w \in v(x)$$
$$j, v \succ x \Longleftrightarrow j \in v(x)^{\triangleright}.$$

- Moreover, for every connective $f(\vec{x}; \vec{y})$ s.t. $\beta(f) = \Diamond$ we set:

- A valuation in a $\mathscr{L}$-frame $\boldsymbol{F}$ is a map $v\colon Var \to \mathcal{G}(W, J, R)$.
- We want to define two relations of satisfaction and co-satisfaction of formulas under v, respectively at worlds $w \in W$ and co-worlds $j \in J$, in symbols

$$w, v \Vdash \varphi \text{ and } j, v \succ \varphi.$$

- For every variable $x \in Var$, we set

$$w, v \Vdash x \iff w \in v(x)$$
$$j, v \succ x \iff j \in v(x)^{\rhd}.$$

- Moreover, for every connective $f(\vec{x}; \vec{y})$ s.t. $\beta(f) = \Diamond$ we set:

$$\begin{aligned} w, v \Vdash f(\vec{\varphi}, \vec{\psi}) \iff & w \in \{r \in W : \text{there are } \vec{u} \in W^m \text{ and } \vec{i} \in J^n \\ & \text{s.t. } \langle \vec{u}, \vec{i}, r\rangle \in T_f \text{ and for all } k \leqslant m, t \leqslant n \\ & u_k, v \Vdash \varphi_k \text{ and } i_t, v \succ \psi_t\}^{\rhd\lhd} \\ j, v \succ f(\vec{\varphi}, \vec{\psi}) \iff & j \in \{w \in W : w, v \Vdash f(\vec{\varphi}, \vec{\psi})\}^{\rhd}. \end{aligned}$$

- A valuation in a $\mathscr{L}$-frame $\boldsymbol{F}$ is a map $v\colon Var \to \mathcal{G}(W, J, R)$.
- We want to define two relations of satisfaction and co-satisfaction of formulas under v, respectively at worlds $w \in W$ and co-worlds $j \in J$, in symbols

$$w, v \Vdash \varphi \text{ and } j, v \succ \varphi.$$

- For every variable $x \in Var$, we set

$$w, v \Vdash x \iff w \in v(x)$$
$$j, v \succ x \iff j \in v(x)^{\rhd}.$$

- Moreover, for every connective $f(\vec{x}; \vec{y})$ s.t. $\beta(f) = \Diamond$ we set:

$$w, v \Vdash f(\vec{\varphi}, \vec{\psi}) \iff w \in \{r \in W : \text{there are } \vec{u} \in W^m \text{ and } \vec{i} \in J^n \text{ s.t. } \langle \vec{u}, \vec{i}, r\rangle \in T_f \text{ and for all } k \leqslant m, t \leqslant n\ u_k, v \Vdash \varphi_k \text{ and } i_t, v \succ \psi_t\}^{\rhd\lhd}$$

$$j, v \succ f(\vec{\varphi}, \vec{\psi}) \iff j \in \{w \in W : w, v \Vdash f(\vec{\varphi}, \vec{\psi})\}^{\rhd}.$$

- A dual definition applied to connectives $f(\vec{x}; \vec{y})$ s.t. $\beta(f) = \Box$.

Frames $\boldsymbol{F}$ can be transformed into algebras $\boldsymbol{F}^{+}$ as follows:

Frames $\boldsymbol{F}$ can be transformed into algebras $\boldsymbol{F}^{+}$ as follows:

- The universe of $\boldsymbol{F}^{+}$ is $\mathcal{G}(W, J, R)$.

Frames $\boldsymbol{F}$ can be transformed into algebras $\boldsymbol{F}^+$ as follows:

- The universe of $\boldsymbol{F}^+$ is $\mathcal{G}(W, J, R)$.
- For every connective $f(z_1, \ldots, z_n)$ and $a_1, \ldots, a_n \in F^+$,

$$f^{\boldsymbol{F}^+}(a_1, \ldots, a_n) := \{w \in W : w, v \Vdash f(z_1, \ldots, z_n)\}$$

where v is any valuation in $\boldsymbol{F}$ s.t. $v(z_i) = a_i$.

Frames $\boldsymbol{F}$ can be transformed into algebras $\boldsymbol{F}^+$ as follows:

- The universe of $\boldsymbol{F}^+$ is $\mathcal{G}(W, J, R)$.
- For every connective $f(z_1, \ldots, z_n)$ and $a_1, \ldots, a_n \in F^+$,

$$f^{\boldsymbol{F}^+}(a_1, \ldots, a_n) \coloneqq \{w \in W : w, v \Vdash f(z_1, \ldots, z_n)\}$$

where v is any valuation in $\boldsymbol{F}$ s.t. $v(z_i) = a_i$.

Definition

Let $\mathscr{L}$ be a labeled ordered language.

Frames $\boldsymbol{F}$ can be transformed into algebras $\boldsymbol{F}^+$ as follows:

- The universe of $\boldsymbol{F}^+$ is $\mathcal{G}(W, J, R)$.
- For every connective $f(z_1, \ldots, z_n)$ and $a_1, \ldots, a_n \in F^+$,

$$f^{\boldsymbol{F}^+}(a_1, \ldots, a_n) \coloneqq \{w \in W : w, v \Vdash f(z_1, \ldots, z_n)\}$$

where v is any valuation in $\boldsymbol{F}$ s.t. $v(z_i) = a_i$.

Definition

Let $\mathscr{L}$ be a labeled ordered language.

1. An $\mathscr{L}$-general frame is a pair $\langle \boldsymbol{F}, A \rangle$ where $\boldsymbol{F}$ is an $\mathscr{L}$-frame and A is the universe of a subalgebra of $\boldsymbol{F}^+$.

Frames $\boldsymbol{F}$ can be transformed into algebras $\boldsymbol{F}^+$ as follows:

- The universe of $\boldsymbol{F}^+$ is $\mathcal{G}(W, J, R)$.
- For every connective $f(z_1, \ldots, z_n)$ and $a_1, \ldots, a_n \in F^+$,

$$f^{\boldsymbol{F}^+}(a_1, \ldots, a_n) := \{w \in W : w, v \Vdash f(z_1, \ldots, z_n)\}$$

  where v is any valuation in $\boldsymbol{F}$ s.t. $v(z_i) = a_i$.

Definition

Let $\mathscr{L}$ be a labeled ordered language.

1. An $\mathscr{L}$-general frame is a pair $\langle \boldsymbol{F}, A \rangle$ where $\boldsymbol{F}$ is an $\mathscr{L}$-frame and A is the universe of a subalgebra of $\boldsymbol{F}^+$.
2. The complex algebra of a general frame $\langle \boldsymbol{F}, A \rangle$ is

$$\langle \boldsymbol{F}, A \rangle^+ := \langle \boldsymbol{A}, \subseteq \rangle \text{ where } \boldsymbol{A} \leqslant \boldsymbol{F}^+.$$

Frames $\boldsymbol{F}$ can be transformed into algebras $\boldsymbol{F}^+$ as follows:

- The universe of $\boldsymbol{F}^+$ is $\mathcal{G}(W, J, R)$.
- For every connective $f(z_1, \dots, z_n)$ and $a_1, \dots, a_n \in F^+$,

$$f^{\boldsymbol{F}^+}(a_1, \dots, a_n) \coloneqq \{w \in W : w, v \Vdash f(z_1, \dots, z_n)\}$$

where v is any valuation in $\boldsymbol{F}$ s.t. $v(z_i) = a_i$.

Definition

Let $\mathscr{L}$ be a labeled ordered language.

1. An $\mathscr{L}$-general frame is a pair $\langle \boldsymbol{F}, A\rangle$ where $\boldsymbol{F}$ is an $\mathscr{L}$-frame and A is the universe of a subalgebra of $\boldsymbol{F}^+$.
2. The complex algebra of a general frame $\langle \boldsymbol{F}, A\rangle$ is

$$\langle \boldsymbol{F}, A\rangle^+ \coloneqq \langle \boldsymbol{A}, \subseteq\rangle \text{ where } \boldsymbol{A} \leqslant \boldsymbol{F}^+.$$

Remark

If $\langle \boldsymbol{F}, A\rangle$ is an $\mathscr{L}$-general frame, then $\langle \boldsymbol{F}, A\rangle^+$ is an $\mathscr{L}$-algebra.

Contents

Let Fr be a class of $\mathscr{L}$-general frames.

Let Fr be a class of $\mathscr{L}$-general frames.

1. The local consequence relation of Fr is:

$$\Gamma \vdash^{l}_{\mathsf{Fr}} \varphi \Longleftrightarrow \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle \in \mathsf{Fr} \text{ and } w \in W$$
$$\text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.$$

Let Fr be a class of $\mathscr{L}$-general frames.

1. The local consequence relation of Fr is:

$$\Gamma \vdash^{l}_{\mathsf{Fr}} \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle \in \mathsf{Fr} \text{ and } w \in W$$
$$\text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.$$

2. The colocal consequence relation of Fr is:

$$\Gamma \vdash^{cl}_{\mathsf{Fr}} \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle \in \mathsf{Fr} \text{ and } j \in J$$
$$\text{if } j, v \succ \Gamma, \text{ then } j, v \succ \varphi.$$

Let Fr be a class of $\mathscr{L}$-general frames.

1. The local consequence relation of Fr is:

$$\Gamma \vdash^{l}_{\mathsf{Fr}} \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A \rangle \in \mathsf{Fr} \text{ and } w \in W$$
$$\text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.$$

2. The colocal consequence relation of Fr is:

$$\Gamma \vdash^{cl}_{\mathsf{Fr}} \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A \rangle \in \mathsf{Fr} \text{ and } j \in J$$
$$\text{if } j, v \succ \Gamma, \text{ then } j, v \succ \varphi.$$

Definition

Let $\mathscr{L}$ be a labeled ordered language. A logic $\vdash$ is a $\mathscr{L}$-local (resp. colocal) consequence if it is the local (resp. colocal) consequence of a class of $\mathscr{L}$-general frames.

Let Fr be a class of $\mathscr{L}$-general frames.

1. The local consequence relation of Fr is:

$$\Gamma \vdash^{l}_{\mathsf{Fr}} \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle \in \mathsf{Fr} \text{ and } w \in W$$
$$\text{if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.$$

2. The colocal consequence relation of Fr is:

$$\Gamma \vdash^{cl}_{\mathsf{Fr}} \varphi \iff \text{for every valuation } v \text{ in } \langle \boldsymbol{F}, A\rangle \in \mathsf{Fr} \text{ and } j \in J$$
$$\text{if } j, v \succ \Gamma, \text{ then } j, v \succ \varphi.$$

Definition

Let $\mathscr{L}$ be a labeled ordered language. A logic $\vdash$ is a $\mathscr{L}$-local (resp. colocal) consequence if it is the local (resp. colocal) consequence of a class of $\mathscr{L}$-general frames.

Remark

A logic is local consequence iff it is a colocal consequence.

Definition

A logic $\vdash$ is monotone if there is an ordered language $\mathscr{L}$ over $\mathscr{L}_{\vdash}$ s.t. every connective $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ is increasing in $\vec{x}$ and decreasing in $\vec{y}$ on $\boldsymbol{Fm}$ w.r.t. $\vdash$,

Definition

A logic $\vdash$ is monotone if there is an ordered language $\mathscr{L}$ over $\mathscr{L}_\vdash$ s.t. every connective $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ is increasing in $\vec{x}$ and decreasing in $\vec{y}$ on $\boldsymbol{Fm}$ w.r.t. $\vdash$, i.e. if for every φ and ψ such that $\varphi \vdash \psi$ we have

$$f(\delta_1, \ldots, \delta_{i-1}, \varphi, \delta_{i+1}, \ldots, \delta_m, \vec{\epsilon}) \vdash f(\delta_1, \ldots, \delta_{i-1}, \psi, \delta_{i+1}, \ldots, \delta_m, \vec{\epsilon})$$

$$f(\vec{\delta}, \epsilon_1, \ldots, \epsilon_{j-1}, \psi, \epsilon_{j+1}, \ldots, \epsilon_n) \vdash f(\vec{\delta}, \epsilon_1, \ldots, \epsilon_{j-1}, \varphi, \epsilon_{j+1}, \ldots, \epsilon_n)$$

for every $\vec{\delta}$ and $\vec{\epsilon}$.

Definition

A logic $\vdash$ is monotone if there is an ordered language $\mathscr{L}$ over $\mathscr{L}_\vdash$ s.t. every connective $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ is increasing in $\vec{x}$ and decreasing in $\vec{y}$ on $\boldsymbol{Fm}$ w.r.t. $\vdash$, i.e. if for every φ and ψ such that $\varphi \vdash \psi$ we have

$$f(\delta_1, \ldots, \delta_{i-1}, \varphi, \delta_{i+1}, \ldots, \delta_m, \vec{\epsilon}) \vdash f(\delta_1, \ldots, \delta_{i-1}, \psi, \delta_{i+1}, \ldots, \delta_m, \vec{\epsilon})$$

$$f(\vec{\delta}, \epsilon_1, \ldots, \epsilon_{j-1}, \psi, \epsilon_{j+1}, \ldots, \epsilon_n) \vdash f(\vec{\delta}, \epsilon_1, \ldots, \epsilon_{j-1}, \varphi, \epsilon_{j+1}, \ldots, \epsilon_n)$$

for every $\vec{\delta}$ and $\vec{\epsilon}$. In this case, $\vdash$ is $\mathscr{L}$-monotone.

Definition

A logic $\vdash$ is monotone if there is an ordered language $\mathscr{L}$ over $\mathscr{L}_\vdash$ s.t. every connective $f(x_1,\ldots,x_m;y_1,\ldots,y_n)$ is increasing in $\vec{x}$ and decreasing in $\vec{y}$ on $\boldsymbol{Fm}$ w.r.t. $\vdash$, i.e. if for every φ and ψ such that $\varphi\vdash\psi$ we have

$$f(\delta_1,\ldots,\delta_{i-1},\varphi,\delta_{i+1},\ldots,\delta_m,\vec{\epsilon})\vdash f(\delta_1,\ldots,\delta_{i-1},\psi,\delta_{i+1},\ldots,\delta_m,\vec{\epsilon})$$
$$f(\vec{\delta},\epsilon_1,\ldots,\epsilon_{j-1},\psi,\epsilon_{j+1},\ldots,\epsilon_n)\vdash f(\vec{\delta},\epsilon_1,\ldots,\epsilon_{j-1},\varphi,\epsilon_{j+1},\ldots,\epsilon_n)$$

for every $\vec{\delta}$ and $\vec{\epsilon}$. In this case, $\vdash$ is $\mathscr{L}$-monotone.

Theorem (Syntactic characterization of local consequences)

Let $\mathscr{L}$ be an ordered language, and β a labeling map. The following conditions are equivalent:

Definition

A logic $\vdash$ is monotone if there is an ordered language $\mathscr{L}$ over $\mathscr{L}_\vdash$ s.t. every connective $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ is increasing in $\vec{x}$ and decreasing in $\vec{y}$ on $\boldsymbol{Fm}$ w.r.t. $\vdash$, i.e. if for every φ and ψ such that $\varphi \vdash \psi$ we have

$$f(\delta_1, \ldots, \delta_{i-1}, \varphi, \delta_{i+1}, \ldots, \delta_m, \vec{\epsilon}) \vdash f(\delta_1, \ldots, \delta_{i-1}, \psi, \delta_{i+1}, \ldots, \delta_m, \vec{\epsilon})$$

$$f(\vec{\delta}, \epsilon_1, \ldots, \epsilon_{j-1}, \psi, \epsilon_{j+1}, \ldots, \epsilon_n) \vdash f(\vec{\delta}, \epsilon_1, \ldots, \epsilon_{j-1}, \varphi, \epsilon_{j+1}, \ldots, \epsilon_n)$$

for every $\vec{\delta}$ and $\vec{\epsilon}$. In this case, $\vdash$ is $\mathscr{L}$-monotone.

Theorem (Syntactic characterization of local consequences)

Let $\mathscr{L}$ be an ordered language, and β a labeling map. The following conditions are equivalent:

1. $\vdash$ is an $\mathscr{L}$-monotone logic.

Definition

A logic $\vdash$ is monotone if there is an ordered language $\mathscr{L}$ over $\mathscr{L}_\vdash$ s.t. every connective $f(x_1,\ldots,x_m;y_1,\ldots,y_n)$ is increasing in $\vec{x}$ and decreasing in $\vec{y}$ on $\boldsymbol{Fm}$ w.r.t. $\vdash$, i.e. if for every φ and ψ such that $\varphi \vdash \psi$ we have

$$f(\delta_1,\ldots,\delta_{i-1},\varphi,\delta_{i+1},\ldots,\delta_m,\vec{\epsilon}) \vdash f(\delta_1,\ldots,\delta_{i-1},\psi,\delta_{i+1},\ldots,\delta_m,\vec{\epsilon})$$

$$f(\vec{\delta},\epsilon_1,\ldots,\epsilon_{j-1},\psi,\epsilon_{j+1},\ldots,\epsilon_n) \vdash f(\vec{\delta},\epsilon_1,\ldots,\epsilon_{j-1},\varphi,\epsilon_{j+1},\ldots,\epsilon_n)$$

for every $\vec{\delta}$ and $\vec{\epsilon}$. In this case, $\vdash$ is $\mathscr{L}$-monotone.

Theorem (Syntactic characterization of local consequences)

Let $\mathscr{L}$ be an ordered language, and β a labeling map. The following conditions are equivalent:

1. $\vdash$ is an $\mathscr{L}$-monotone logic.
2. $\vdash$ is an $\mathscr{L}^\beta$-local consequence.

Contents

Definition

Let $\vdash$ be a logic and $\mathscr{L}$ be an ordered language over $\mathscr{L}_{\vdash}$.

Definition

Let $\vdash$ be a logic and $\mathscr{L}$ be an ordered language over $\mathscr{L}_{\vdash}$.

1. An $\mathscr{L}$-algebra $\langle \boldsymbol{A}, \leqslant \rangle$ is an $\mathscr{L}$-ordered model of $\vdash$ if for every $a \in A$ the upset $\uparrow a$ is a deductive filter of $\vdash$.

Definition

Let $\vdash$ be a logic and $\mathscr{L}$ be an ordered language over $\mathscr{L}_{\vdash}$.

1. An $\mathscr{L}$-algebra $\langle \boldsymbol{A}, \leqslant \rangle$ is an $\mathscr{L}$-ordered model of $\vdash$ if for every $a \in A$ the upset $\uparrow a$ is a deductive filter of $\vdash$.
2. Accordingly, we set

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash) := \{\langle \boldsymbol{A}, \leqslant \rangle : \langle \boldsymbol{A}, \leqslant \rangle \text{ is an } \mathscr{L}\text{-ordered model of } \vdash\}.$$

Definition

Let $\vdash$ be a logic and $\mathscr{L}$ be an ordered language over $\mathscr{L}_{\vdash}$.

1. An $\mathscr{L}$-algebra $\langle \boldsymbol{A}, \leqslant \rangle$ is an $\mathscr{L}$-ordered model of $\vdash$ if for every $a \in A$ the upset $\uparrow a$ is a deductive filter of $\vdash$.
2. Accordingly, we set

$$\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash) := \{\langle \boldsymbol{A}, \leqslant \rangle : \langle \boldsymbol{A}, \leqslant \rangle \text{ is an } \mathscr{L}\text{-ordered model of } \vdash\}.$$

Remark

$\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is closed under $\mathbb{S}$ and $\mathbb{P}$ (and $\mathbb{P}_{\mathsf{u}}$ if $\vdash$ is finitary).

Definition

Let $\vdash$ be a logic and $\mathscr{L}$ be an ordered language over $\mathscr{L}_{\vdash}$.

1. An $\mathscr{L}$-algebra $\langle \boldsymbol{A}, \leqslant \rangle$ is an $\mathscr{L}$-ordered model of $\vdash$ if for every $a \in A$ the upset $\uparrow a$ is a deductive filter of $\vdash$.
2. Accordingly, we set

$$\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash) := \{\langle \boldsymbol{A}, \leqslant \rangle : \langle \boldsymbol{A}, \leqslant \rangle \text{ is an } \mathscr{L}\text{-ordered model of } \vdash\}.$$

Remark

$\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is closed under $\mathbb{S}$ and $\mathbb{P}$ (and $\mathbb{P}_{\mathsf{u}}$ if $\vdash$ is finitary).

- Non-mathematical thesis: $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ should be understood as the class of distinguished ordered models of $\vdash$ (from the point of view of the ordered language $\mathscr{L}$).

Theoretic justification of $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$

Definition

Let $\vdash$ be a logic and $\langle \boldsymbol{F}, A \rangle$ be an $\mathscr{L}$-general frame.

Theoretic justification of $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$

Definition

Let $\vdash$ be a logic and $\langle \boldsymbol{F}, A\rangle$ be an $\mathscr{L}$-general frame.

1. $\langle \boldsymbol{F}, A\rangle$ is a model of $\vdash$ if its local consequence extends $\vdash$.

Theoretic justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$

Definition

Let $\vdash$ be a logic and $\langle \boldsymbol{F}, A \rangle$ be an $\mathscr{L}$-general frame.

1. $\langle \boldsymbol{F}, A \rangle$ is a model of $\vdash$ if its local consequence extends $\vdash$.
2. $\langle \boldsymbol{F}, A \rangle$ is a co-model of $\vdash$ if its co-local consequence extends $\vdash$.

Theoretic justification of $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$

Definition

Let $\vdash$ be a logic and $\langle \boldsymbol{F}, A \rangle$ be an $\mathscr{L}$-general frame.

1. $\langle \boldsymbol{F}, A \rangle$ is a model of $\vdash$ if its local consequence extends $\vdash$.
2. $\langle \boldsymbol{F}, A \rangle$ is a co-model of $\vdash$ if its co-local consequence extends $\vdash$.

Theorem

Let $\vdash$ be a logic, $\mathscr{L}$ an ordered lang. over $\mathscr{L}_{\vdash}$, β a labeling map.

Theoretic justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$

Definition

Let $\vdash$ be a logic and $\langle \boldsymbol{F}, A\rangle$ be an $\mathscr{L}$-general frame.

1. $\langle \boldsymbol{F}, A\rangle$ is a model of $\vdash$ if its local consequence extends $\vdash$.
2. $\langle \boldsymbol{F}, A\rangle$ is a co-model of $\vdash$ if its co-local consequence extends $\vdash$.

Theorem

Let $\vdash$ be a logic, $\mathscr{L}$ an ordered lang. over $\mathscr{L}_{\vdash}$, β a labeling map.

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash) = \{\langle \boldsymbol{F}, A\rangle^{+} : \langle \boldsymbol{F}, A\rangle \text{ is an } \mathscr{L}^{\beta}\text{-general frame and a model of } \vdash\}.$$

Theoretic justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$

Definition

Let $\vdash$ be a logic and $\langle \boldsymbol{F}, A\rangle$ be an $\mathscr{L}$-general frame.

1. $\langle \boldsymbol{F}, A\rangle$ is a model of $\vdash$ if its local consequence extends $\vdash$.
2. $\langle \boldsymbol{F}, A\rangle$ is a co-model of $\vdash$ if its co-local consequence extends $\vdash$.

Theorem

Let $\vdash$ be a logic, $\mathscr{L}$ an ordered lang. over $\mathscr{L}_{\vdash}$, β a labeling map.

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash) = \{\langle \boldsymbol{F}, A\rangle^{+} : \langle \boldsymbol{F}, A\rangle \text{ is an } \mathscr{L}^{\beta}\text{-general frame and a model of } \vdash\}.$$

In other words, $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is the class of complex algebras of relational models of $\vdash$ (from the point of view of $\mathscr{L}$ and β).

Theoretic justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$

Definition

Let $\vdash$ be a logic and $\langle \boldsymbol{F}, A\rangle$ be an $\mathscr{L}$-general frame.

1. $\langle \boldsymbol{F}, A\rangle$ is a model of $\vdash$ if its local consequence extends $\vdash$.
2. $\langle \boldsymbol{F}, A\rangle$ is a co-model of $\vdash$ if its co-local consequence extends $\vdash$.

Theorem

Let $\vdash$ be a logic, $\mathscr{L}$ an ordered lang. over $\mathscr{L}_{\vdash}$, β a labeling map.

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash) = \{\langle \boldsymbol{F}, A\rangle^{+} : \langle \boldsymbol{F}, A\rangle \text{ is an } \mathscr{L}^{\beta}\text{-general frame and a model of } \vdash\}.$$

In other words, $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is the class of complex algebras of relational models of $\vdash$ (from the point of view of $\mathscr{L}$ and β).

- Rephrasing: Logics may have a semantics of ordered algebras, because they have a local relational semantics.

Empiric justification of $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: semilattice-based logics

Theorem

Let K be a variety with a semilattice reduct s.t. when ordered under the meet-order is a class of $\mathscr{L}$-algebras.

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: semilattice-based logics

Theorem

Let K be a variety with a semilattice reduct s.t. when ordered under the meet-order is a class of $\mathscr{L}$-algebras. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}}) = \{\langle \boldsymbol{A}, \leqslant\rangle : \boldsymbol{A} \in \mathsf{K} \text{ and } \leqslant \text{ is the meet-order of } \boldsymbol{A}\}.$$

Empiric justification of $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: semilattice-based logics

Theorem

Let K be a variety with a semilattice reduct s.t. when ordered under the meet-order is a class of $\mathscr{L}$-algebras. Then

$$\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}}) = \{\langle \boldsymbol{A}, \leqslant\rangle : \boldsymbol{A} \in \mathsf{K} \text{ and } \leqslant \text{ is the meet-order of } \boldsymbol{A}\}.$$

Examples:

Empiric justification of $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: semilattice-based logics

Theorem

Let K be a variety with a semilattice reduct s.t. when ordered under the meet-order is a class of $\mathscr{L}$-algebras. Then

$$\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}}) = \{\langle \boldsymbol{A}, \leqslant\rangle : \boldsymbol{A} \in \mathsf{K} \text{ and } \leqslant \text{ is the meet-order of } \boldsymbol{A}\}.$$

Examples:

- Let K be a variety of modal algebras, and $\vdash$ the local consequence of the normal modal logic associated with K.

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: semilattice-based logics

Theorem

Let K be a variety with a semilattice reduct s.t. when ordered under the meet-order is a class of $\mathscr{L}$-algebras. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}}) = \{\langle \boldsymbol{A}, \leqslant\rangle : \boldsymbol{A} \in \mathsf{K} \text{ and } \leqslant \text{ is the meet-order of } \boldsymbol{A}\}.$$

Examples:

- Let K be a variety of modal algebras, and $\vdash$ the local consequence of the normal modal logic associated with K. Then $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is K with the lattice order (for the natural $\mathscr{L}$).

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: semilattice-based logics

Theorem

Let K be a variety with a semilattice reduct s.t. when ordered under the meet-order is a class of $\mathscr{L}$-algebras. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}}) = \{\langle \boldsymbol{A}, \leqslant\rangle : \boldsymbol{A} \in \mathsf{K} \text{ and } \leqslant \text{ is the meet-order of } \boldsymbol{A}\}.$$

Examples:

- Let K be a variety of modal algebras, and $\vdash$ the local consequence of the normal modal logic associated with K. Then $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is K with the lattice order (for the natural $\mathscr{L}$).
- Let K be a variety of Heyting algebras, and $\vdash$ the superintuitionistic logic associated with K.

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: semilattice-based logics

Theorem

Let K be a variety with a semilattice reduct s.t. when ordered under the meet-order is a class of $\mathscr{L}$-algebras. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}}) = \{\langle \boldsymbol{A}, \leqslant\rangle : \boldsymbol{A} \in \mathsf{K} \text{ and } \leqslant \text{ is the meet-order of } \boldsymbol{A}\}.$$

Examples:

- Let K be a variety of modal algebras, and $\vdash$ the local consequence of the normal modal logic associated with K. Then $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is K with the lattice order (for the natural $\mathscr{L}$).
- Let K be a variety of Heyting algebras, and $\vdash$ the superintuitionistic logic associated with K. Then $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$ is K with the lattice order (for the natural $\mathscr{L}$).

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: intensional fragments

For the natural ordered languages $\mathscr{L}$:

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: intensional fragments

For the natural ordered languages $\mathscr{L}$:

- Let $\mathbf{IPC}_{\rightarrow}$ be the $\langle\rightarrow\rangle$-fragment of Intuitionistic Logic.

Empiric justification of $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: intensional fragments

For the natural ordered languages $\mathscr{L}$:

- Let $\mathbf{IPC}_{\to}$ be the $\langle\to\rangle$-fragment of Intuitionistic Logic. Then

$$\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\to}) = \text{ Hilbert algebras } + \text{Hilbert-order.}$$

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: intensional fragments

For the natural ordered languages $\mathscr{L}$:

- Let $\mathbf{IPC}_{\rightarrow}$ be the $\langle\rightarrow\rangle$-fragment of Intuitionistic Logic. Then

$$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\rightarrow}) = \text{ Hilbert algebras} + \text{Hilbert-order.}$$

- Let $\mathbf{InFL}^{\leqslant}_{\mathsf{e}}$ be the $\langle\cdot,\rightarrow\rangle$-fragment of the logic preserving degrees of truth of commutative FL-algebras.

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: intensional fragments

For the natural ordered languages $\mathscr{L}$:

- Let $\mathbf{IPC}_{\to}$ be the $\langle \to \rangle$-fragment of Intuitionistic Logic. Then

  $$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\to}) = \text{ Hilbert algebras} + \text{Hilbert-order.}$$

- Let $\mathbf{InFL}^{\leqslant}_{\mathsf{e}}$ be the $\langle \cdot, \to \rangle$-fragment of the logic preserving degrees of truth of commutative FL-algebras. Then

  $$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}^{\leqslant}_{\mathsf{e}}) = \langle \cdot, \to, \leqslant \rangle\text{-subreducts of commutative FL-algebras.}$$

Empiric justification of $\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: intensional fragments

For the natural ordered languages $\mathscr{L}$:

- Let $\mathbf{IPC}_{\to}$ be the $\langle \to \rangle$-fragment of Intuitionistic Logic. Then

  $$\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\to}) = \text{ Hilbert algebras} + \text{Hilbert-order.}$$

- Let $\mathbf{InFL}^{\leqslant}_{\mathsf{e}}$ be the $\langle \cdot, \to \rangle$-fragment of the logic preserving degrees of truth of commutative FL-algebras. Then

  $$\mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}^{\leqslant}_{\mathsf{e}}) = \langle \cdot, \to, \leqslant \rangle\text{-subreducts of commutative FL-algebras.}$$

- Let $\mathbf{InR}^{\leqslant}$ be the $\langle \cdot, \to, \neg \rangle$-fragment of the logic preserving degrees of truth of De Morgan monoids.

Empiric justification of $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash)$: intensional fragments

For the natural ordered languages $\mathscr{L}$:

- Let $\mathbf{IPC}_{\rightarrow}$ be the $\langle\rightarrow\rangle$-fragment of Intuitionistic Logic. Then

  $$\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\rightarrow}) = \text{ Hilbert algebras} + \text{Hilbert-order.}$$

- Let $\mathbf{InFL}^{\leqslant}_{\mathbf{e}}$ be the $\langle\cdot,\rightarrow\rangle$-fragment of the logic preserving degrees of truth of commutative FL-algebras. Then

  $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}^{\leqslant}_{\mathbf{e}}) = \langle\cdot,\rightarrow,\leqslant\rangle$-subreducts of commutative FL-algebras.

- Let $\mathbf{InR}^{\leqslant}$ be the $\langle\cdot,\rightarrow,\neg\rangle$-fragment of the logic preserving degrees of truth of De Morgan monoids. Then

  $\mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InR}^{\leqslant}) = \langle\cdot,\rightarrow,\neg,\leqslant\rangle$-subreducts of De Morgan monoids.

Contents

1. What is a frame? (for an arbitrary algebraic language)

2. What does it mean that a logic has a local relational semantics?

3. Why do most logics have a semantics of ordered algebras?

4. Are there logic-based dualities/completions for ordered algebras?

Logics preserving degrees of truth of Lattice Expansions

Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras.

Logics preserving degrees of truth of Lattice Expansions

Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras. Then for all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}})$ we have:

Logics preserving degrees of truth of Lattice Expansions

Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras. Then for all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}})$ we have:

1. $\boldsymbol{A} \in \mathsf{K}$ and $\leqslant$ is the lattice order of $\boldsymbol{A}$.

Logics preserving degrees of truth of Lattice Expansions

Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras. Then for all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}})$ we have:

1. $\boldsymbol{A} \in \mathsf{K}$ and $\leqslant$ is the lattice order of $\boldsymbol{A}$.
2. $\mathrm{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ lattice filters and } J = \text{ lattice ideals.}$$

Logics preserving degrees of truth of Lattice Expansions

Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras. Then for all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}})$ we have:

1. $\boldsymbol{A} \in \mathsf{K}$ and $\leqslant$ is the lattice order of $\boldsymbol{A}$.
2. $\mathsf{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ lattice filters and } J = \text{ lattice ideals.}$$

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_{+})^{+}$ is the canonical extension of $\langle \boldsymbol{A}, \leqslant \rangle$.

Logics preserving degrees of truth of Lattice Expansions

Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras. Then for all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}})$ we have:

1. $\boldsymbol{A} \in \mathsf{K}$ and $\leqslant$ is the lattice order of $\boldsymbol{A}$.
2. $\mathsf{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ lattice filters and } J = \text{ lattice ideals.}$$

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_+)^+$ is the canonical extension of $\langle \boldsymbol{A}, \leqslant \rangle$.

Implicative fragment of **IPC**

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\rightarrow})$ we have:

Logics preserving degrees of truth of Lattice Expansions

Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras. Then for all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}})$ we have:

1. $\boldsymbol{A} \in \mathsf{K}$ and $\leqslant$ is the lattice order of $\boldsymbol{A}$.
2. $\mathsf{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ lattice filters and } J = \text{ lattice ideals.}$$

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_+)^+$ is the canonical extension of $\langle \boldsymbol{A}, \leqslant \rangle$.

Implicative fragment of IPC

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\rightarrow})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a Hilbert algebra equipped with the Hilbert-order.

Logics preserving degrees of truth of Lattice Expansions

Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras. Then for all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}})$ we have:

1. $\boldsymbol{A} \in \mathsf{K}$ and $\leqslant$ is the lattice order of $\boldsymbol{A}$.
2. $\mathsf{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$W =$ lattice filters and $J =$ lattice ideals.

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_+)^+$ is the canonical extension of $\langle \boldsymbol{A}, \leqslant \rangle$.

Implicative fragment of **IPC**

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\rightarrow})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a Hilbert algebra equipped with the Hilbert-order.
2. $\mathsf{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$W =$ implicative filters and $J =$ downsets.

Logics preserving degrees of truth of Lattice Expansions

Let K be a variety with a bounded lattice reduct s.t. when ordered under the lattice-order is a class of $\mathscr{L}$-algebras. Then for all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\vdash^{\leqslant}_{\mathsf{K}})$ we have:

1. $\boldsymbol{A} \in \mathsf{K}$ and $\leqslant$ is the lattice order of $\boldsymbol{A}$.
2. $\mathsf{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$W =$ lattice filters and $J =$ lattice ideals.

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_{+})^{+}$ is the canonical extension of $\langle \boldsymbol{A}, \leqslant \rangle$.

Implicative fragment of IPC

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{IPC}_{\rightarrow})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a Hilbert algebra equipped with the Hilbert-order.
2. $\mathsf{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$W =$ implicative filters and $J =$ downsets.

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_{+})^{+}$ is intrinsically a Heyting algebra.

Intensional fragment of $\mathbf{FL}_{\mathsf{e}}^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}_{\mathscr{L}}^{\leqslant}(\mathbf{InFL}_{\mathsf{e}}^{\leqslant})$ we have:

Intensional fragment of $\mathbf{FL}_e^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}_e^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \leqslant \rangle$-subreduct of a commutative FL-algebra.

Intensional fragment of $\mathbf{FL}_{\mathrm{e}}^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}_{\mathrm{e}}^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \leqslant \rangle$-subreduct of a commutative FL-algebra.
2. $\mathrm{Pol}_{\mathscr{L}} \langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ upsets and } J = \text{ downsets}.$$

Intensional fragment of $\mathbf{FL}_e^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}_e^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \leqslant \rangle$-subreduct of a commutative FL-algebra.
2. $\mathsf{Pol}_{\mathscr{L}} \langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ upsets and } J = \text{ downsets}.$$

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_+)^+$ is intrinsically a commutative FL-algebra.

Intensional fragment of $\mathbf{FL}_e^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant\rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}_e^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant\rangle$ is a $\langle \cdot, \to, \leqslant\rangle$-subreduct of a commutative FL-algebra.
2. $\mathrm{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant\rangle = \langle W, J, R\rangle$ is s.t.

$$W = \text{ upsets and } J = \text{ downsets.}$$

Moreover, $(\langle \boldsymbol{A}, \leqslant\rangle_+)^+$ is intrinsically a commutative FL-algebra.

Intensional fragment of $\mathbf{R}^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant\rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InR}^{\leqslant})$ we have:

Intensional fragment of $\mathbf{FL}_{\mathsf{e}}^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}_{\mathsf{e}}^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \leqslant \rangle$-subreduct of a commutative FL-algebra.
2. $\mathsf{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ upsets and } J = \text{ downsets.}$$

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_{+})^{+}$ is intrinsically a commutative FL-algebra.

Intensional fragment of $\mathbf{R}^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathsf{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InR}^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \neg, \leqslant \rangle$-subreduct of a De Morgan monoid.

Intensional fragment of $\mathbf{FL}_e^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}_e^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \leqslant \rangle$-subreduct of a commutative FL-algebra.
2. $\mathrm{Pol}_{\mathscr{L}} \langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ upsets and } J = \text{ downsets.}$$

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_+)^+$ is intrinsically a commutative FL-algebra.

Intensional fragment of $\mathbf{R}^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InR}^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \neg, \leqslant \rangle$-subreduct of a De Morgan monoid.
2. $\mathrm{Pol}_{\mathscr{L}} \langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ intensional filters and } J = \text{ intensional ideals.}$$

Intensional fragment of $\mathbf{FL}_e^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InFL}_e^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \leqslant \rangle$-subreduct of a commutative FL-algebra.
2. $\mathrm{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ upsets and } J = \text{ downsets.}$$

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_+)^+$ is intrinsically a commutative FL-algebra.

Intensional fragment of $\mathbf{R}^{\leqslant}$

For all $\langle \boldsymbol{A}, \leqslant \rangle \in \mathrm{Alg}^{\leqslant}_{\mathscr{L}}(\mathbf{InR}^{\leqslant})$ we have:

1. $\langle \boldsymbol{A}, \leqslant \rangle$ is a $\langle \cdot, \to, \neg, \leqslant \rangle$-subreduct of a De Morgan monoid.
2. $\mathrm{Pol}_{\mathscr{L}}\langle \boldsymbol{A}, \leqslant \rangle = \langle W, J, R \rangle$ is s.t.

$$W = \text{ intensional filters and } J = \text{ intensional ideals.}$$

Moreover, $(\langle \boldsymbol{A}, \leqslant \rangle_+)^+$ is intrinsically a De Morgan monoid.

A sample of what comes next...

- Substructural logics with weakening can be viewed as global consequences in this spirit,

A sample of what comes next...

- Substructural logics with weakening can be viewed as global consequences in this spirit, e.g. Łukasiewicz is the global version of the logic preserving degrees of truth of MV-algebras.

A sample of what comes next...

- Substructural logics with weakening can be viewed as global consequences in this spirit, e.g. Łukasiewicz is the global version of the logic preserving degrees of truth of MV-algebras.
- One can give a relational semantics for every logic, inspired by the Routley-Meyer semantics for Relevance Logic.

A sample of what comes next...

- Substructural logics with weakening can be viewed as global consequences in this spirit, e.g. Łukasiewicz is the global version of the logic preserving degrees of truth of MV-algebras.
- One can give a relational semantics for every logic, inspired by the Routley-Meyer semantics for Relevance Logic.
- We can delete co-worlds from frames in nice cases, e.g. distributive substructural and modal logics.

A sample of what comes next...

- Substructural logics with weakening can be viewed as global consequences in this spirit, e.g. Łukasiewicz is the global version of the logic preserving degrees of truth of MV-algebras.
- One can give a relational semantics for every logic, inspired by the Routley-Meyer semantics for Relevance Logic.
- We can delete co-worlds from frames in nice cases, e.g. distributive substructural and modal logics.
- This approach suggests a semantic-based of expanding every local consequence to the first-order lever with quantifiers and identity, which is axiomatized very transparently by means of meta-rules.

A sample of what comes next...

- Substructural logics with weakening can be viewed as global consequences in this spirit, e.g. Łukasiewicz is the global version of the logic preserving degrees of truth of MV-algebras.
- One can give a relational semantics for every logic, inspired by the Routley-Meyer semantics for Relevance Logic.
- We can delete co-worlds from frames in nice cases, e.g. distributive substructural and modal logics.
- This approach suggests a semantic-based of expanding every local consequence to the first-order lever with quantifiers and identity, which is axiomatized very transparently by means of meta-rules.
- This yields a complete alternative relational semantics for all first-order modal and superintuitionistic logics.

A sample of what comes next...

- Substructural logics with weakening can be viewed as global consequences in this spirit, e.g. Łukasiewicz is the global version of the logic preserving degrees of truth of MV-algebras.
- One can give a relational semantics for every logic, inspired by the Routley-Meyer semantics for Relevance Logic.
- We can delete co-worlds from frames in nice cases, e.g. distributive substructural and modal logics.
- This approach suggests a semantic-based of expanding every local consequence to the first-order lever with quantifiers and identity, which is axiomatized very transparently by means of meta-rules.
- This yields a complete alternative relational semantics for all first-order modal and superintuitionistic logics.

...thank you for coming!